Global Aspects of Holonomy in Pseudo-Riemannian Geometry

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Abstract

A pseudo-Riemannian vector bundle (E, h, ∇, π, X) is a smooth real vector bundle $\pi : E \to X$ with a bundle metric h of signature (r, s) on E and a metric connection ∇ on (E, h). Suppose the full holonomy group $Hol(\nabla) \subset O(r, s)$ acts weakly irreducibly and let $W \neq \{0\}$ be an isotropic $Hol(\nabla)$-invariant subspace such that $\dim \tilde{W} \leq \dim W =: r$ for each isotropic invariant subspace $\tilde{W}$. The holonomy principle implies the existence of a ∇-invariant isotropic subbundle Ξ or rank r, i.e., we derive a pseudo-Riemannian vector bundle structure $(\mathcal{S}, h^{\mathcal{S}}, \nabla^{\mathcal{S}}, \pi^{\mathcal{S}}, X)$ on the quotient $\mathcal{S} := \mathrm{Coker}(\Xi \hookrightarrow \Xi^{\perp})$. We refer to this bundle as the *screen bundle* of (E, h, ∇, π, X) and call $Hol(\nabla^{\mathcal{S}})$ the *screen holonomy* of (E, h, ∇, π, X).

Screen bundles naturally appear in two situations which are considered in this thesis: As the screen bundle of the tangent bundle of a Lorentzian manifold whose holonomy acts weakly irreducibly or as the screen bundle of the normal bundle of a non-degenerate submanifold of a pseudo-Riemannian space of constant curvature.
Chapter 1 contains an introduction to all concepts used in this presentation.
In Chapter 2 we first introduce weakly irreducible, reducible Lorentzian metrics on the total spaces of certain S^1-bundles. Using Hodge theory we provide sufficient conditions under which the screen holonomy is Hermitian or flat. In particular, we construct examples with disconnected Hermitian screen holonomy.

A weakly irreducible, reducible, time-orientable Lorentzian manifold naturally admits a codimension one foliation $\mathcal{X}^{\perp}$ and a subfoliation of dimension one. Using Riemannian foliation theory the subsequent section focuses on the interplay of Lorentzian data and the topology of the leaves. In particular, we classify holonomy representations for those X for which $\mathcal{X}^{\perp}$ admits a compact leaf with finite fundamental group.

Finally, we introduce a Bochner technique for any Lorentzian manifold X admitting a parallel lightlike vector field such that all leaves of $\mathcal{X}^{\perp}$ are compact. We prove that if the (transverse) Ricci curvature of X is non-negative the first Betti number of X is bounded by $1 \leq b_1 \leq \dim X$ if X is compact and $0 \leq b_1 \leq \dim X - 1$ otherwise. Moreover, we show that these bounds are optimal using the Lorentzian manifolds constructed in the first part and obtain further results depending on the screen holonomy of X.

Chapter 3 primarily focuses on the classification of holonomy groups of the normal bundle of submanifolds in spaces of constant curvature. In the first section we extend Olmos' well known classification result to spacelike submanifolds in Lorentzian spaces of constant curvature by showing that the normal screen holonomy is that of a Riemannian symmetric space.

In general, the normal screen bundle of a non-degenerate submanifold in a pseudo-Riemannian space of constant curvature may again admit a screen bundle and so on. We organize this data in a finite rooted tree to which we refer as the *screen tree* and define the class of (very) good submanifolds for which we classify the leaves of the screen tree. In the final section we study tubes along subbundles which are basically restrictions of the tubular neighborhood map to a subbundle of the normal bundle. As an application we derive a construction generating a submanifold with irreducible screen holonomy from a submanifold with non-degenerately reducible screen holonomy. In particular, we provide a method to construct very good submanifolds from certain good submanifolds.

Zusammenfassung

Sei (E, h, ∇, π, X) pseudo-Riemannsches Vektorbündel. Falls die volle Holonomiegruppe $Hol(\nabla) \subset O(r,s)$ schwach irreduzibel wirkt und ein isotroper, $Hol(\nabla)$-invarianter Unterraum $W \neq \{0\}$ existiert, so dass $\dim \tilde{W} \leq \dim W =: r$ für jeden Unterraum $\tilde{W}$ mit diesen Eigenschaften gilt, so impliziert das Holonomieprinzip die Existenz eines ∇-invarianten, isotropen, rang r Unterbündels Ξ, und man erhält eine pseudo-Riemannsche Vektorbündelstruktur $(\mathcal{S}, h^{\mathcal{S}}, \nabla^{\mathcal{S}}, \pi^{\mathcal{S}}, X)$ auf dem Quotienten $\mathcal{S} := \mathrm{Coker}(\Xi \hookrightarrow \Xi^{\perp})$. Dieses Bündel bezeichnen wir als Schirmbündel und nennen $Hol(\nabla^{\mathcal{S}})$ die Schirmholonomie von (E, h, ∇, π, X). Schirmbündel treten u.a. als Schirmbündel des Tagentialbündels einer Lorentzmannigfaltigkeit oder als Schirmbündel eines Normalenbündels einer nicht-degenerierten Untermannigfaltigkeit in einem pseudo-Riemannschen Raum konstanter Krümmung auf.

Kapitel 1 beinhaltet eine Einführung zu allen Konzepten, die in dieser Präsentation benutzt werden.

In Kapitel 2 führen wir zunächst schwach irreduzible Lorentzmetriken auf den Totalräumen von gewissen S^1-Bündeln ein. Mittels Hodge-Theorie leiten wir hinreichende Bedingungen her, unter denen die Schirmholonomie hermitesch oder flach ist. Auf einer schwach irreduziblen, reduziblen, zeitorientierbaren Lorentzmannigfaltigkeit existiert eine kanonische Blätterung $\mathcal{X}^{\perp}$ der Codimension 1 und eine Unterblätterung der Dimension 1. Im anschließenden Abschnitt untersuchen wir das Zusammenspiel von Lorentzdaten und der Topologie der Blätter mittels Riemannscher Blätterungstheorie. Insbesondere klassifizieren wir die Holonomiedarstellungen von Lorentzmannigfaltigkeiten bei denen $\mathcal{X}^{\perp}$ ein kompaktes Blatt mit endlicher Fundamentalgruppe hat. Schließlich führen wir eine Bochner Technik für jede Lorentzmannigfaltigkeit X mit einem parallelen, lichtartigen Vektorfeld ein, bei der $\mathcal{X}^{\perp}$ ausschließlich kompakte Blätter hat. Falls X eine nicht-negative (transversale) Ricci-Krümmung hat, so zeigen wir, dass die erste Bettizahl von X durch $1 \leq b_1 \leq \dim X$ (X kompakt) bzw. $0 \leq b_1 \leq \dim X - 1$ (X nicht kompakt) beschränkt wird. Mit Hilfe der oben beschriebenen Konstruktion zeigen wir, dass diese Schranken optimal sind und leiten in Abhängigkeit von der Schirmholonomie weitere Resultate her.

In Kapitel 3 beschäftigen wir uns mit der Klassifikation von Holonomiegruppen des Normalenbündels von Untermannigfaltigkeiten in Räumen konstanter Krümmung. Im ersten Abschnitt erweitern wir Olmos' hinlänglich bekanntes Klassifikationsresultat auf raumartige Untermannigfaltigkeiten in Lorentzmannigfaltigkeiten konstanter Krümmung, indem wir zeigen, dass die normale Schirmholonomie die eines Riemannsch symmetrischen Raumes ist. Im Allg. hat das normale Schirmbündel einer nicht-degenerierten Untermannigfaltigkeit in einem pseudo-Riemannschen Raum konstanter Krümmung wieder ein Schirmbündel usw. Wir organisieren diese Daten in einem endlichen, gewurzelten Baum, welchen wir Schirmbaum nennen und definieren die Klasse der (sehr) guten Untermannigfaltigkeiten, für die wir die Blätter des Schirmbaums klassifizieren. Im letzten Abschnitt betrachten wir Tuben entlang von Unterbündeln welche wir als Einschränkung der Tubenabbildung auf ein Unterbündel des Normalenbündels auffassen können. Als Anwendung erhalten wir eine Konstruktion, die eine Untermannigfaltigkeit mit irreduzibler Schirmholonomie aus einer Untermannigfaltigkeit mit nicht-degeneriert reduzibler Schirmholonomie erzeugt. Insbesondere leiten wir eine Methode her, mit der man sehr gute Untermannigfaltigkeiten aus gewissen guten Untermannigfaltigkeiten erzeugen kann.

Contents

Introduction

Let $E \to X$ be a smooth real vector bundle with a bundle metric h of signature $(1, n)$ for which ∇^E is a metric connection and suppose the action $Hol(\nabla^E) \subset O(1, n)$ is reducible, but does not admit any non-degenerate invariant subspace. Such a bundle naturally appears in two situations. As the tangent bundle of a weakly irreducible, reducible Lorentzian manifold with its Levi-Civita connection and as the normal bundle of certain submanifolds in pseudo-Riemannian spaces of constant curvature with its induced connection $\nabla^\perp$. The purpose of this thesis is to study the geometry of E in both cases.

A. Motivation and Open Problems.

1. Let us first consider tangent bundles. A famous problem in physics is to find exact solutions to Einstein's field equation $Ric - \frac{1}{2}\mathrm{scal} \cdot g^L = 8\pi T$, where T is the stress-energy tensor. As a model for electromagnetic and gravitational radiation Lorentzian metrics of the local form

$$ds^2 = 2dudv - H(u, x^i)du^2 + \sum_i (dx^i)^2$$

were proposed (cf. [Kun61]) and some authors refer to these spacetimes as *pp-waves*. All metrics of the above form admit the parallel lightlike vector field $\frac{\partial}{\partial v}$. In general, Lorentzian spacetimes admitting a parallel lightlike vector field are locally of the Walker form [Wal50]

$$ds^2 = 2dudv - H(u, x^i)du^2 + 2A_j(u, x^i)dudx^j + g_{ij}dx^j dx^k$$

and such spacetimes have been proposed as solutions of supergravity (cf. [SSJ03] and the references therein).

2. In addition to the motivations implied by physics there are also purely mathematical reasons to study Lorentzian manifolds admitting a parallel lightlike vector field. In order to understand this we consider a Lorentzian manifold (X, g^L) and its holonomy algebra $\mathfrak{hol}_p(X, g^L)$ at $p \in X$. It is a well known result that $\mathfrak{hol}_p(X, g^L)$ is a Berger algebra, i.e.,

$$\mathfrak{hol}_p(X, g^L) = \mathrm{span}\{R(x, y) : R \in \mathcal{K}(\mathfrak{hol}_p),\ x, y \in T_pX\},$$

where $\mathcal{K}(\mathfrak{g}) := \{R \in \Lambda^2 T_p^* X \otimes \mathfrak{g} : R(x, y)z + R(y, z)x + R(z, x)y = 0\}$. Thus, there is an orthogonal decomposition

$$T_pX = E_0 \oplus \ldots \oplus E_\ell$$

into non-degenerate $\mathfrak{hol}_p(X, g^L)$-invariant subspaces E_j and a corresponding decomposition

$$\mathfrak{hol}_p(X, g^L) = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_\ell$$

into commuting ideals such that each $\mathfrak{h}_j$ acts weakly irreducibly on E_j and trivially on E_i for $i \neq j$. In particular, $\mathfrak{hol}_p(X, g^L)$ acts trivially on E_0 and we may assume w.l.o.g. that either E_0 or E_1 is not positive definite.

Suppose $E_1 \neq \{0\}$ has Lorentzian signature and $\mathfrak{h}_1 \neq \mathfrak{so}(1, \dim E_1 - 1)$. We will explain in Section 2.3 that the full holonomy group $Hol(X, g^L)$ admits a lightlike invariant line. Moreover, there exists a nowhere vanishing lightlike vector field $V \in \Gamma(X, TX)$ such that $\nabla^L_\cdot V = \alpha(\cdot)V$ for some 1-form $\alpha \in \Gamma(X, T^*X)$ if (X, g^L) is time-orientable. For the Lie algebra $\mathfrak{h}_1 \subset \mathfrak{so}(1, q+1)$ acting on E_1 there is the following deep

Theorem 0.1 (Bérard-Bergery & Ikemakhen, Leistner)**.** *For any weakly irreducible, reducible Lie algebra $\mathfrak{h}_1 \subset \mathfrak{so}(1, q+1)$ there exists $\mathfrak{g} \subset \mathfrak{so}(q)$ such that $\mathfrak{h}_1$ belongs to one of the following types:*

- *Type 1:* $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^q$
- *Type 2:* $\mathfrak{h} = \mathfrak{g} \ltimes \mathbb{R}^q$
- *Type 3:*
$$\mathfrak{h} = \left\{ \begin{pmatrix} \varphi(A) & w^T & 0 \\ 0 & A & -w \\ 0 & 0 & -\varphi(A) \end{pmatrix} : A \in \mathfrak{g},\ w \in \mathbb{R}^q \right\}$$
where $\varphi : \mathfrak{g} \twoheadrightarrow \mathbb{R}$ is an epimorphism satisfying $\varphi|_{[\mathfrak{g},\mathfrak{g}]} = 0$.
- *Type 4: There is $0 < \ell < q$ such that $\mathbb{R}^q = \mathbb{R}^\ell \oplus \mathbb{R}^{q-\ell}$, $\mathfrak{g} \subset \mathfrak{so}(\ell)$ and*
$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & \psi(A)^T & w^T & 0 \\ 0 & 0 & 0 & -\psi(A) \\ 0 & 0 & A & -w \\ 0 & 0 & 0 & 0 \end{pmatrix} : A \in \mathfrak{g},\ w \in \mathbb{R}^\ell \right\}$$
for some epimorphism $\psi : \mathfrak{g} \twoheadrightarrow \mathbb{R}^{q-\ell}$ satisfying $\psi|_{[\mathfrak{g},\mathfrak{g}]} = 0$.

Moreover, if $\mathfrak{h}_1$ is a component of a Lorentzian holonomy algebra then $\mathfrak{g}$ acts as a Riemannian holonomy representation. □

For simplicity, suppose $\mathfrak{hol}_p(X, g^L) = \mathfrak{h}_1$. Since $Hol_p(X, g^L)$ leaves a lightlike line $\Xi_p \subset T_pX$ invariant the holonomy principle implies the existence of a vector bundle Ξ induced by Ξ_p. Since Ξ is a subbundle of its orthogonal complement $\Xi^\perp$ we have a quotient $\mathcal{S} := \mathrm{Coker}(\Xi \hookrightarrow \Xi^\perp)$ to which we refer as the *screen bundle* of (X, g^L). The Lorentzian metric g^L naturally induces a Riemannian bundle metric on $\mathcal{S}$.

Moreover, the Levi-Civita connection ∇^L induces a connection $\nabla^{\mathcal{S}}$ on $\mathcal{S}$ which is metric w.r.t. the Riemannian bundle metric.

It was first proved by Leistner [Lei06] that $\mathfrak{hol}_p(\nabla^{\mathcal{S}}) = \mathfrak{g}$ where $\mathfrak{g}$ is the algebra from the above theorem. Therefore, we call $\mathfrak{g}$ the *screen holonomy* of (X, g^L). Finally, $\Xi^\perp$ induces a foliation $\mathcal{X}^\perp$ of codimension one on X and there is a 1-dimensional subfoliation $\mathcal{X}$ of $\mathcal{X}^\perp$ on X which is induced by Ξ. These observations motivate the following highly non-trivial questions which are the major reason for the existence of this thesis.

Problem 0.2. How does the geometry resp. topology of (X, g^L) restrict $\mathfrak{hol}(X, g^L)$ and $\mathfrak{g}$?

Problem 0.3. How do $\mathfrak{hol}_p(X, g^L)$ and $\mathfrak{g}$ restrict the geometry resp. topology of (X, g^L)?

The most difficult part in the classification of Lorentzian holonomy algebras was to prove that $\mathfrak{g}$ acts as a Riemannian holonomy representation. However, Leistner's proof in [Lei07] is solely based on representation theory and does not provide geometric reasons for the result. Therefore, we have the technical

Problem 0.4. Find a geometric proof for the classification of screen holonomy representations.

3. Let us focus on the normal bundle next. We say that a smooth map $f : (X, g) \to (Y, h)$ between pseudo-Riemannian manifolds is a submanifold if $f^*h = g$. We may consider TX as a subbundle of $TY|_X := f^*TY$ and define the normal bundle NX of f as the h-orthogonal complement of $TX \subset TY|_X$. The Levi-Civita connection ∇^Y of (Y, h) induces a metric connection $\nabla^\perp$ on $(NX, h|_{NX})$.

   In the following we suppose that $(Y, h =: \langle\cdot,\cdot\rangle)$ is a pseudo-Riemannian space of constant curvature. Since any pseudo-Riemannian manifold admits an isometric embedding into a space of constant curvature it is natural to ask for a classification of holonomy representations of $\nabla^\perp$ for such submanifolds. In case that h is positive definite this has been achieved in [Olm90]. In fact, we have

   Theorem 0.5 (Olmos)**.** *Let $f : (X, g) \to (Y, \langle\cdot,\cdot\rangle)$ be a submanifold and let $\mathfrak{hol}_p(\nabla^\perp)$ be the normal holonomy algebra at $p \in X$. If $(Y, \langle\cdot,\cdot\rangle)$ is a Riemannian manifold of constant curvature then there exists an orthogonal $\mathfrak{hol}_p(\nabla^\perp)$-invariant decomposition $N_pX = E_0 \oplus \ldots \oplus E_\ell$ and a decomposition*

   $$\mathfrak{hol}_p(\nabla^\perp) = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_\ell$$

   into ideals such that $\mathfrak{h}_i$ acts as the holonomy representation of an irreducible Riemannian symmetric space on E_i and trivially on E_j for $i \neq j$. □

   Galaev has proved in [Gal04] that there is no classification of holonomy representations of pseudo-Riemannian manifolds of signature $(r, \dim X - r)$ if $r > 1$. Therefore, we propose the following

   Problem 0.6. Let $f : (X, g) \to (Y, \langle\cdot,\cdot\rangle)$ be a spacelike submanifold in a pseudo-Riemannian space of constant curvature.

- Classify normal holonomy representations if $(Y, \langle\cdot,\cdot\rangle)$ is of Lorentzian signature.
- Is there a classification of normal holonomy representations if $(Y, \langle\cdot,\cdot\rangle)$ is neither Riemannian nor Lorentzian?
- In which way does the classification extend to arbitrary non-degenerate submanifolds?

In order to give an answer to the first problem if $\mathfrak{hol}_p(\nabla^\perp)$ acts weakly irreducibly, but reducibly we will define the normal screen bundle $\mathcal{S}$ and classify its holonomy representations. As for Lorentzian manifolds the holonomy of $\mathcal{S}$ can be non-degenerately reducible and there is no geometric de Rham decomposition theorem for $\mathcal{S}$, i.e., we derive the technical

Problem 0.7. Given a spacelike submanifold $f : (X, g) \to (Y, \langle\cdot,\cdot\rangle)$ in a Lorentzian space of constant curvature whose normal screen holonomy is non-degenerately reducible, is there a method to modify f in order to derive a submanifold with irreducible normal screen holonomy?

B. Outline of the Thesis.

This presentation is divided into three chapters and one appendix.

1. The first chapter is intended to provide an introduction to all techniques applied in this thesis. We do not seek completeness or elegance of the exposition as none of the stated results in unknown. However, we supply enough details to make the text accessible to readers not familiar with one of the techniques and refer to further literature for more comprehensive introductions.

   The first section summarizes the holonomy theory for connections on a vector bundle. As long as one is concerned with the classification of holonomy groups of the tangent or the normal bundle the techniques are very similar. Hence, we use the notion of pseudo-Riemannian vector bundles in order to keep the exposition short. A pseudo-Riemannian vector bundle (E, h, ∇, π, X) is a smooth real vector bundle $\pi : E \to X$ with a bundle metric h of signature (r, s) on E and a metric connection ∇ on (E, h) and we say (E, h, ∇, π, X) has good holonomy if $\mathfrak{hol}_p(\nabla)$ is a Berger algebra.

   Once we study submanifolds we must deal with different signatures. Therefore, we say that given a pseudo-Euclidean vector space (E, h) a subgroup $G \subset O(E, h)$ acts weakly irreducibly with index r if there is no proper non-degenerate G-invariant subspace of E, but there is a G-invariant isotropic subspace $\Xi \subset E$ such that any isotropic G-invariant subspace $\tilde{\Xi}$ satisfies $\dim \tilde{\Xi} \leq \dim \Xi = r$.

   In the second section we present necessary results on Kähler and holomorphic symplectic manifolds. In particular, we review the description of S^1-bundles by Čech cohomology and its relation to the complex structure. These results are applied to construct Lorentzian manifolds with Hermitian screen holonomy.

In the third section we review some results on Riemannian foliations and basic Hodge theory. These techniques allow us to introduce a Bochner technique as explained below.

2. The main purpose of the second chapter is to study Lorentzian manifolds admitting a holonomy invariant lightlike line. The first section introduces screen bundles in a general way in order to be applicable to submanifolds as well. Given a pseudo-Riemannian vector bundle (E, h, ∇^E, π, X) whose full holonomy group $Hol(\nabla^E) \subset O(p, 2r-p+q)$ is weakly irreducible with index r where $1 \leq r \leq p$ and $1 \leq q \geq 2(p-r)$ we derive an isotropic ∇^E-parallel subbundle $\Xi \subset E$.[1] Its orthogonal complement $\Xi^\perp \subset E$ is again ∇^E-parallel. As above, we derive a pseudo-Riemannian vector bundle $(\mathcal{S}, h^{\mathcal{S}}, \nabla^S, \pi^S, X)$ where $\mathcal{S} := \mathrm{Coker}(\Xi \hookrightarrow \Xi^\perp)$ to which we refer as the screen bundle of (E, h, ∇^E, π, X).

   More generally, suppose (E, h, ∇^E, π, X) is a pseudo-Riemannian vector bundle such that X is simply connected. A *screen tree* of (E, h, ∇^E, π, X) is a finite rooted tree which is derived as follows: The root is given by $\mathfrak{hol}(\nabla^E)$ and if there is a Borel-Lichnérowicz decomposition $\mathfrak{hol}(\nabla^E) = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_\ell$ then each irreducible $\mathfrak{h}_i$ is defined to be a child of the root. By the algebraic de Rham decomposition each $\mathfrak{h}_i$ corresponds to a pseudo-Riemannian vector subbundle and if $\mathfrak{h}_i$ acts weakly irreducibly with positive index we may consider its screen bundle $\mathcal{S}_i$. If $\mathcal{S}_i$ is of rank zero then we attach a trivial vertex. Otherwise we attach the holonomy algebra of $\mathcal{S}_i$ to the root.

   Then we proceed by induction and define any irreducible or trivial vertex and any vertex not admitting a Borel-Lichnérowicz decomposition to be a leaf. Moreover, a screen tree is said to be complete if all leaves are either irreducible or trivial.

   If $Hol_p(\nabla^E) \subset O(p, 2r-p+q)$ is weakly irreducible with index r then we can find a basis $(v_1, \ldots, v_r, e_1, \ldots, e_q, w_1, \ldots, w_r)$ of E_p such that $v_i \in \Xi_p$, $e_i \in \Xi_p^\perp$, $\langle v_i, w_j \rangle = \delta_{ij}$, $\langle e_i, e_j \rangle = \varepsilon_i \delta_{ij}$ and $\langle e_i, w_j \rangle = 0$. Then the orthogonal part $G = pr_{O(p-r,q-p+r)}(Hol(\nabla^E))$ of $Hol(\nabla^E)$ is defined by restriction and projection to $\{e_i\}$ and we will prove that $Hol(\nabla^{\mathcal{S}}) = G$ explaining the importance of the screen bundle. For the rest of the first section we study the relation of the full holonomy of ∇^E to its algebra.

   Consider a Lorentzian manifold (X, g^L) admitting a nowhere vanishing lightlike vector field V such that $\nabla^E_\cdot V = \alpha(\cdot)V$ for some 1-form $\alpha \in \Gamma(X, TX)$. As above we have the foliations $\mathcal{X}^\perp$ and $\mathcal{X}$ on X and in light of Problem 0.2 it is natural to ask which holonomy groups are possible if the leaves of $\mathcal{X}$ are compact. Although the answer is simple if $(X = S^1 \times S^1 \times M, g^L = 2dxdz + f dz^2 + g_M)$ nothing is known if $\mathcal{X}$ is a non-trivial foliation. Hence, we introduce the following construction in the second section.

[1] We focus w.l.o.g. on subalgebras in $\mathfrak{so}(r, s)$ where $r \leq s$. Hence, we need the inequality $q \geq 2(p-r)$ in order to ensure $2r - p + q \geq p$.

Proposition 0.8 ([Lär08a]). *Let (M,g) be an $(n+1)$-dimensional Riemannian manifold and η a nowhere vanishing closed 1-form on M. Let ψ be a 2-form on M such that $[\frac{\psi}{2\pi}] \in H^2(M,\mathbb{Z})$. Then there exists an S^1-bundle $\pi : X \to M$ satisfying $c_1(X \to M) = -[\frac{\psi}{2\pi}]$ and*

a) There is a global nowhere vanishing 1-form θ on X such that

$$\tilde{g}_f := 2\theta\pi^*\eta + f \cdot \pi^*\eta^2 + \pi^* g$$

defines a Lorentzian metric on X for any $f \in C^\infty(X)$.

b) Given $p \in X$ and a local 1-form ϕ with $\psi = d\phi$ there are local coordinates $(x, y^1, \dots, y^n, z)$ around p such that

$$\tilde{g}_f = 2dxdz + 2(u_i + g_{i(n+1)})dy^i dz + (f + 2u_{n+1} + g_{(n+1)(n+1)})dz^2 + g_{ij}dy^i dy^j$$

where $\phi = u_i dy^i + u_{n+1}dz$.

c) The $U(1)$-action of $X \to M$ acts by isometries on $(X, \tilde{g}_f)$ if f is constant on the fibers.

d) The vertical vector field is a global lightlike vector field which is parallel if and only if f is constant on the fibers.

□

If X is constructed as in the proposition the foliation $\mathcal{X}$ is induced by the $U(1)$-action on X. We say that $f \in C^\infty(X)$ is *suitable* if $\mathfrak{hol}(X, \tilde{g}_f)$ is weakly irreducible and not of type 4 in Thm. 0.1. We will provide a construction for suitable functions and prove that $\mathfrak{hol}(X, \tilde{g}_f)$ is never of type 3 and of type 1 if and only if $\frac{\partial f}{\partial x}|_p \neq 0$ for some $p \in X$. Then we proceed to study the screen holonomy of (X, g^L) by focusing on the following special case: Given a Riemannian manifold (N, g) and the principal S^1-bundle $\tilde{X} \to N$ corresponding to $-[\frac{\psi}{2\pi}] \in H^2(N,\mathbb{Z})$ we consider $X = \tilde{X} \times L$ with the metric $\tilde{g}_f$ where $L \in \{\mathbb{R}, S^1\}$ and $\eta = dz$ is the coordinate 1-form on L. In this case, we say $(X, \tilde{g}_f)$ is of *toric type* over (N, g).

The advantage of toric type manifolds is that we can identify the screen bundle with the horizontal distribution $\mathrm{Ker}(\theta)$ in TX and prove that $Hol(N,g) \subset Hol(\nabla^{\mathcal{S}})$. In order to show the converse inclusion we consider the lift $\tilde{T}$ of (N,g)-parallel tensors on TN to $\mathcal{S}$ and examine the equation $\nabla^{\mathcal{S}}_{\cdot}\tilde{T} = 0$.

We will see that this equation is equivalent to a condition on the representative ψ of $[\psi]$. If $Hol(N,g)$ is Hermitian and N is compact the equation for ψ implies a Hodge theoretic condition on its cohomology class. Then we apply Hodge theory to solve the equation on the level of cohomology classes.

If ψ is the harmonic representative of $[\psi]$ on (N,g) the Hodge theorem ensures that ψ solves the equation on the level of forms. This way, we construct weakly irreducible Lorentzian manifolds with Hermitian screen holonomy for which all leaves of $\mathcal{X}$ are compact. E.g., we will prove

Proposition 0.9. *Let (M,J) be a simple holomorphic symplectic manifold of complex dimension $2n$ such that $\rho(M,J) = b_2(M) - 2$ and $b_2(M) \geq 4$.*[2] *Then there exists a hyperkähler structure $(J_1 = J, J_2, J_3, g)$ with Kähler class $[\omega] \in H^2(M,\mathbb{Z})$ on M and $0 \neq [\frac{\psi}{2\pi}] \in H^{1,1}(M,J) \cap H^{1,1}(M,J_2) \cap H^2(M,\mathbb{Z})$. Moreover, if $(X, \tilde{g}_f)$ is of toric type over (M,J,g) where $c_1(\tilde{X} \to M) = -[\frac{\psi}{2\pi}]$ and $\tilde{g}_f$ is constructed using the harmonic representative ψ of $[\psi]$ with $f \in C^\infty(X)$ suitable then*

$$Hol(X, \tilde{g}_f) = \begin{cases} Sp(n) \ltimes \mathbb{R}^{4n} & \text{if } \frac{\partial f}{\partial x} \equiv 0, \\ (\mathbb{R}^* \times Sp(n)) \ltimes \mathbb{R}^{4n} & \text{otherwise.} \end{cases}$$

□

In fact, a more substantial application of Hodge theory allows us to construct disconnected Hermitian screen holonomies, e.g.,

Proposition 0.10. *Let (M,J) be an Enriques surface with Ricci-flat Kähler metric g and Kähler form ω such that $[\omega] \in H^2(M,\mathbb{Z})$. Then there exists $0 \neq [\frac{\psi}{2\pi}] \in H^{1,1}_{prim}((M,J,[\omega]),\mathbb{Z})$ and if $(X = \tilde{X} \times L, \tilde{g}_f)$ is of toric type over (M,J,g) where $c_1(\tilde{X} \to M) = -[\frac{\psi}{2\pi}]$ and $\tilde{g}_f$ is constructed using the harmonic representative ψ of $[\psi]$ with $f \in C^\infty(X)$ suitable then there is a disconnected subgroup $G \subset U(2)$ whose identity component is $SU(2)$ such that*

$$Hol(X, \tilde{g}_f) = \begin{cases} G \ltimes \mathbb{R}^{4} & \text{if } \frac{\partial f}{\partial x} \equiv 0, \\ (\mathbb{R}^* \times G) \ltimes \mathbb{R}^{4} & \text{otherwise.} \end{cases}$$

□

In light of the motivations from physics we say that a weakly irreducible Lorentzian manifold is a *pp-wave* if $\nabla^{\mathcal{S}}$ is flat. In the last part of the second section we construct compact toric type pp-waves such that $\mathcal{X}$ is a non-trivial foliation with compact leaves.

In the third section we focus on Problem 0.3 and 0.4. Given a Lorentzian manifold (X, g^L) and a global nowhere vanishing lightlike vector field $V \in \Gamma(X, TX)$ we say (X, g^L, V) is *almost decent* if $\nabla^L_\cdot = \alpha(\cdot)V$ for some 1-form $\alpha \in \Gamma(X, T^*X)$. Almost decent spacetimes are not required to be weakly irreducible, but we still have the foliations $\mathcal{X}^\perp$ and $\mathcal{X}$ as above. Moreover, we define the screen bundle of (X, g^L, V) in the same way as for weakly irreducible Lorentzian manifolds and say that (X, g^L, V) is *decent* if furthermore $\alpha|_{\Xi^\perp} = 0$.

Given a splitting $s : \mathcal{S} \to \Xi^\perp$ of the exact sequence

$$0 \longrightarrow \Xi \longrightarrow \Xi^\perp \longrightarrow \mathcal{S} \longrightarrow 0$$

we say that $S := s(\mathcal{S}) \subset TX$ is a (non-canonical) realization of the screen bundle in TX. Each realization S of the screen bundle uniquely defines a lightlike vector

[2] We write $\rho(M,J)$ for the Picard number of (M,J).

field $Z \in \Gamma(X, TX)$ such that $g^L(V, Z) = 1$ as well as $g^L(S, Z) = 0$. In particular, the Levi-Civita connection of (X, g^L) induces a connection ∇^S on S.

A Riemannian metric g^R on an arbitrary foliated manifold $(X, \mathcal{F})$ is said to be *bundle-like* if $(L_W g^R)(Y_1, Y_2) = 0$ for any $W \in \Gamma(X, T\mathcal{F})$ and all $Y_1, Y_2 \in \Gamma(X, T\mathcal{F}^{\perp_{g^R}})$. If (X, g^L, V) is almost decent and S is any realization of the screen bundle we define

$$g^R(A, B) := \begin{cases} 1 & \text{if } A = B = V \text{ or } A = B = Z, \\ g^L(A, B) & \text{if } A, B \in S, \\ 0 & \text{if } A \in S,\ B \in \{V, Z\} \text{ or } A = V,\ B = Z. \end{cases}$$

Given a leaf $\mathcal{L}^\perp$ of $\mathcal{X}^\perp$ the metric $g^R|_{\mathcal{L}^\perp}$ is bundle-like w.r.t. $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ and g^R is bundle-like w.r.t. $(X, \mathcal{X}^\perp)$ if (X, g^L, V) is decent. There is a transverse Levi-Civita connection associated to any Riemannian foliation and the key feature of g^R is then given by

Proposition 0.11. *Let (X, g^L, V) be an almost decent spacetime and $\mathcal{L}^\perp$ a leaf of $\mathcal{X}^\perp$. For any realization S of the screen bundle the transverse Levi-Civita connection of $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp}, g^R|_{\mathcal{L}^\perp})$ coincides with $\nabla^S|_{\mathcal{L}^\perp}$.* □

Given an almost decent spacetime (X, g^L, V) and a realization S of the screen bundle we say that (X, g^L, V, S) is *almost horizontal* if $[V, Y] \in S$ for any local section $Y \in \Gamma(U, S)$. An almost horizontal spacetime is horizontal if it is decent. The Lorentzian manifolds from the second section are horizontal and although toric type Lorentzian manifolds appear to be very special we can prove

Theorem 0.12. *Let (X, g^L, V, S) be a horizontal spacetime where $Z \in \Gamma(X, TX)$ is complete. Suppose $\mathcal{L}^\perp$ is a leaf of $\mathcal{X}^\perp$ and write $\tilde{X}$ for the universal cover of X. If all leaves of $\mathcal{X}|_{\mathcal{L}^\perp}$ are compact with trivial leaf holonomy then $\tilde{X}$ is diffeomorphic to the universal cover of a toric type Lorentzian manifold. Moreover, if $\mathcal{L}^\perp$ is closed in X then X is covered by a toric type Lorentzian manifold.* □

Then we proceed to analyze the interplay of various Lorentzian causality conditions and Riemannian foliation theory. E.g., we can prove

Theorem 0.13. *Let (X, g^L, V) be a simply connected, causal, decent spacetime which is lightlike complete. Let $\mathcal{L}^\perp$ be any leaf of $\mathcal{X}^\perp$ and S a realization of the screen bundle. Suppose $Z \in \Gamma(X, TX)$ is complete and one of the following conditions holds:*

- *$g^R|_{\mathcal{L}^\perp}$ is complete and (X, g^L) is strongly causal at $p \in X$,*
- *(X, g^L, V, S) is horizontal and strongly causal.*

Then $\mathcal{L}^\perp = \mathbb{R} \times M$ and $X = \mathbb{R}^2 \times M$ where $M := \mathcal{L}^\perp / \mathcal{X}|_{\mathcal{L}^\perp}$ is a smooth manifold. □

The topology of M in the preceding Theorem is in fact restricted by $Hol(\nabla^S)$. E.g., we derive

Corollary 0.14. *Let (X, g^L, V) be a simply connected, lightlike complete, causal, decent spacetime such that $\dim_{\mathbb{R}} X = 10$ and $b_6(X) = 1$. Let $\mathcal{L}^\perp$ be a leaf of $\mathcal{X}^\perp$ and suppose there is a realization S of the screen bundle such that $g^R|_{\mathcal{L}^\perp}$ and $Z \in \Gamma(X, TX)$ are complete.*

If $Hol(\nabla^S|_{\mathcal{L}^\perp}) \subset \{0\} \times SU(3)$ then $X = \mathbb{R}^4 \times M$ where M is a simply connected compact manifold admitting a Ricci-flat Kähler metric. □

Stably causal almost decent spacetimes are shown to admit an integrable realization of the screen bundle. For that reason, we briefly study decent spacetimes admitting such a realization. In particular, we show that a toric type manifold over M does not have an integrable realization provided M is compact simply connected and $0 \neq [\psi] \in H^2(M, \mathbb{R})$.

In the final part of the third section we study holonomy representations of almost decent spacetimes in light of Problem 0.4. The idea is to analyze the structure of spacetimes for which Simons' theory of holonomy systems fails and to exclude these. We obtain

Theorem 0.15. *Let (X, g^L) be a time-orientable Lorentzian manifold such that $\mathfrak{hol}(X, g^L)$ acts weakly irreducibly with index 1 and suppose the associated foliation $\mathcal{X}^\perp$ admits a compact leaf $\mathcal{L}^\perp$ such that $\pi_1(\mathcal{L}^\perp)$ is finite. Then $\mathfrak{hol}(X, g^L)$ belongs to one of the following types where $\mathfrak{g} := \mathfrak{hol}(\nabla^S)$.*

- *Type 1:* $\mathfrak{hol}(X, g^L) = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^{\dim X - 2}$
- *Type 2:* $\mathfrak{hol}(X, g^L) = \mathfrak{g} \ltimes \mathbb{R}^{\dim X - 2}$
- *Type 3:*

$$\mathfrak{hol}(X, g^L) = \left\{ \begin{pmatrix} \varphi(A) & w^T & 0 \\ 0 & A & -w \\ 0 & 0 & -\varphi(A) \end{pmatrix} : A \in \mathfrak{g},\ w \in \mathbb{R}^{\dim X - 2} \right\}$$

where $\varphi : \mathfrak{g} \twoheadrightarrow \mathbb{R}$ is an epimorphism satisfying $\varphi|_{[\mathfrak{g},\mathfrak{g}]} = 0$.

Moreover, identifying $\mathfrak{g} \subset \mathfrak{so}(\dim X - 2)$ there are decompositions

$$\mathbb{R}^{\dim X - 2} = F_1 \oplus \ldots \oplus F_\ell \quad \text{and} \quad \mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_\ell$$

such that each $\mathfrak{g}_j$ acts trivially on F_i for $i \neq j$ and as an irreducible Riemannian holonomy representation on F_j. In particular, $\mathfrak{g}$ does not act trivially on any subspace of $\mathbb{R}^{\dim X - 2}$. □

In the fourth section we introduce a Bochner technique allowing to do curvature comparison on decent spacetimes. Any Bochner technique needs some kind of compactness condition as it involves a Hodge theorem. Decent spacetimes for which the leaves of $\mathcal{X}$ are compact are already studied in the third section. Hence, we require the leaves of $\mathcal{X}^\perp$ to be compact. Then we prove the Bochner technique in three steps. First, we use a Mayer-Vietoris argument to relate the cohomology of X to that of a leaf $\mathcal{L}^\perp$ of $\mathcal{X}^\perp$. Then we apply a Gysin sequence to compute the cohomology of $\mathcal{L}^\perp$ in terms of the basic cohomology of the foliated manifold $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$. Finally, the basic cohomology is related to the curvature of (X, g^L) by a Weitzenböck formula. In particular, we can prove

Theorem 0.16 ([Lär10])**.** *Let (X, g^L, V) be a decent spacetime and $\mathcal{L}^\perp$ a leaf of $\mathcal{X}^\perp$. Suppose $Ric^L(W, W) \geq 0$ for all $W \in T\mathcal{L}^\perp$.*

a) If X is compact and $\mathcal{X}^\perp$ admits a compact leaf then $1 \leq b_1(X) \leq \dim X$.

b) If X is non-compact and all leaves of $\mathcal{X}^\perp$ are compact then $0 \leq b_1(X) \leq \dim X - 1$.

Moreover, if $Ric^L_q(W, W) > 0$ for some $q \in \mathcal{L}^\perp$ and all $W \in S_q$ the bounds are $1 \leq b_1(X) \leq 2$ and $0 \leq b_1(X) \leq 1$ respectively. □

Then we apply toric type Lorentzian manifolds once again to show optimality of the bounds in the preceding Theorem. Moreover, we relate the inequality $Ric^L(W, W) \geq 0$ to the *strong energy condition.* Finally, the holonomy of the screen bundle provides further bounds on the Betti numbers as we will see in

Proposition 0.17. *Let (X, g^L, V) be a decent spacetime and $\mathcal{L}^\perp$ a compact leaf of $\mathcal{X}^\perp$. Suppose $Hol(\nabla^S|_{\mathcal{L}^\perp})$ is irreducible and $Ric^L(W, W) \geq 0$ for all $W \in T\mathcal{L}^\perp$.*

a) If X is compact then $b_1(X) \in \{1, 2\}$ and $b_2(X) \leq \dim H^2_B(\mathcal{X}|_{\mathcal{L}^\perp}) + 1$.

b) If X is non-compact and all leaves of $\mathcal{X}^\perp$ are compact then $b_1(X) \in \{0, 1\}$ and $b_2(X) \in \{\dim H^2_B(\mathcal{X}|_{\mathcal{L}^\perp}) - 1, \dim H^2_B(\mathcal{X}|_{\mathcal{L}^\perp})\}$.

Moreover, if $Hol(\nabla^S|_{\mathcal{L}^\perp}) = SU(n)$ with $n \geq 3$ we can replace $H^2_B(\mathcal{X}|_{\mathcal{L}^\perp})$ by $H^{1,1}_B(\mathcal{X}|_{\mathcal{L}^\perp})$ and if $Hol(\nabla^S|_{\mathcal{L}^\perp}) = Sp(n)$ with $n \geq 1$ we can replace $\dim H^2_B(\mathcal{X}|_{\mathcal{L}^\perp})$ by $\dim H^{1,1}_B(\mathcal{X}|_{\mathcal{L}^\perp}) + 2$.[3] □

3. The first section of the third chapter is intended to provide an answer to Problem 0.6. Given a submanifold $f : (X, g) \to (Y, h)$ in a Riemannian space of constant curvature the key idea in the proof of Olmos' theorem 0.5 is to define an algebraic curvature tensor $\mathcal{R}_p$ with non-vanishing scalar curvature on (N_pX, h) such that

$$\mathfrak{hol}_p(\nabla^\perp) = \mathrm{span}\{\mathcal{R}_p^{\tau_\gamma^\perp}(\tau_\gamma^\perp\xi, \tau_\gamma^\perp\eta) : \gamma : [0,1] \to X,\ \gamma(0) = p,\ \xi, \eta \in N_pX\}$$

where $\mathcal{R}_p^{\tau_\gamma^\perp} = {\tau_\gamma^\perp}^{-1} \circ \mathcal{R}_{\gamma(1)} \circ \tau_\gamma^\perp$ and $\tau_\gamma^\perp$ is the parallel displacement along γ w.r.t. $\nabla^\perp$. This idea does not extend to arbitrary pseudo-Riemannian submanifolds. Therefore, we introduce

Definition 0.18. *Let $f : (X, g) \to (Y, \langle\cdot,\cdot\rangle)$ be a submanifold of signature (r, s) in a space of constant curvature $(Y, \langle\cdot,\cdot\rangle)$ and let $R^\perp_p$ be its normal curvature tensor. Define $K_p := Ker(R^\perp_p \circ F^{-1})$, where $F : \Lambda^2 T_pX \to \mathfrak{so}(T_pX, g)$ is the natural isomorphism. Then, we say X is a*

[3] As for simple holomorphic symplectic manifolds, if $Hol(\nabla^S|_{\mathcal{L}^\perp}) = Sp(n)$ the dimension of $H^{1,1}_B(\mathcal{X}|_{\mathcal{L}^\perp})$ coincides for each of the three associated complex structures.

- *good submanifold if for all $p \in X$ the subspace $K_p \subset \mathfrak{so}(T_pX, g)$ is non-degenerate w.r.t. the Killing form on $\mathfrak{so}(T_pX, g)$.*
- *very good submanifold if for all $p \in X$ the subspace $K_p^\perp \subset \mathfrak{so}(T_pX, g)$ is definite w.r.t. the Killing form on $\mathfrak{so}(T_pX, g)$.*

Using a slight modification of Olmos' idea the pseudo-Riemannian vector bundle $(NX, h|_{NX}, \nabla^\perp, \pi, X)$ is easily shown to have good holonomy in the above sense if f is a good submanifold. However, the classification of normal holonomy representations is not completed at this point since the Borel-Lichnérowicz decomposition of $\mathfrak{hol}_p(\nabla^\perp)$ might admit weakly irreducible, reducible components. Hence, we proceed to study the screen holonomy algebras of all weakly irreducible components and if f is a very good submanifold we can prove that the screen holonomy algebra is a Berger algebra. Inductively, we show

Theorem 0.19. *Let $f : (X, g) \to (Y, \langle\cdot,\cdot\rangle)$ be a simply connected very good submanifold in a space of constant curvature. Then*

- *$(NX, h|_{NX}, \nabla^\perp, \pi, X)$ admits a complete screen tree whose non-trivial leaves are given by Berger's list (Thm. 1.9),*
- *any non-trivial leaf given by a representation on a definite space acts as the holonomy representation of an irreducible Riemannian symmetric space.* $\square$

Combining Thm. 0.19 and Thm. 0.1 we obtain a complete classification for the normal holonomy of spacelike submanifolds in Lorentzian spaces of constant curvature (cf. [Lär08b]). Note, that this case was already studied in [OW01]. In fact, Olmos and Will proved that the restricted normal holonomy group acts polarly. However, in light of Thm. 0.1 this does not imply any restrictions on $\mathfrak{g} \subset \mathfrak{so}(q)$, i.e., Thm. 0.19 is an improvement.

In the second section of the third chapter we focus on Problem 0.7. For simplicity, we restrict to embedded submanifolds in $\mathbb{R}^{r,s}$ and given a subbundle $E \subset NX$ its intersection with a tubular neighborhood of X provides a new submanifold $\mathcal{U}_E$.

Then the idea is to consider subbundles E given by "disturbing directions". As an application we can construct non-spacelike very good submanifolds:

Proposition 0.20. *Let $f : (X, g) \to (\mathbb{R}^{r+s}, \langle\cdot,\cdot\rangle_{r,s})$ be an embedded good submanifold whose normal holonomy acts weakly irreducibly with index 1. If S is a realization of the normal screen bundle then there is an open subset $\tilde{X} \supset X$ of $\mathcal{U}_{S^\perp} \subset (\mathbb{R}^{r+s}, \langle\cdot,\cdot\rangle_{r,s})$ which is a very good submanifold if and only if for all $p \in X$ the subspace $G_p(\Lambda^2 S_p) \subset \mathfrak{so}(T_pX, g)$ is definite w.r.t. the Killing form.*

Moreover, the normal holonomy representation of $\tilde{X} \subset (\mathbb{R}^{r+s}, \langle\cdot,\cdot\rangle_{r,s})$ contains that of the normal screen bundle. $\square$

On the other hand, we have

Proposition 0.21. *Let $f : (X, g) \to (\mathbb{R}^{r+s}, \langle\cdot,\cdot\rangle_{r,s})$ be a simply connected embedded very good submanifold whose normal holonomy acts weakly irreducibly with index 1. Let S be a non-canonical realization of the normal screen bundle and $\mathfrak{g}$ its holonomy representation at $p \in X$.*

If $S_p = S_1|_p \oplus S_2|_p$ is a $\mathfrak{g}$-invariant orthogonal decomposition into non-degenerate subspaces such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_i$ acts trivially on $S_j|_p$ for $i \neq j$ and $\mathfrak{hol}_p(\nabla^{\perp,f})$ contains the ideal $\mathbb{R}^{\operatorname{codim} X-2}$, i.e.,

$$\left\{ \begin{pmatrix} 0 & w^T & 0 \\ 0 & 0 & -w \\ 0 & 0 & 0 \end{pmatrix} : w \in \mathbb{R}^{\operatorname{codim} X-2} \right\} \subset \mathfrak{hol}_p(\nabla^{\perp,f}) \subset \mathfrak{so}(S_p) \ltimes \mathbb{R}^{\operatorname{codim} X-2}$$

then

- *there is an open subset $\tilde{X} \supset X$ of $\mathcal{U}_{S_2} \subset (\mathbb{R}^{r+s}, \langle\cdot,\cdot\rangle_{r,s})$ which is a very good submanifold, where S_2 is the subbundle corresponding to $S_2|_p$,*
- *$\mathfrak{hol}_p(\nabla^{\perp,\tilde{X}})$ acts weakly irreducibly with index 1,*
- *the normal screen bundle of $\tilde{X}$ extends the bundle S_1 corresponding to $S_1|_p$ and the normal screen holonomy of $\nabla^{\perp,\tilde{X}}$ is given by $\mathfrak{g}_1$.* □

Finally, we obtain further applications with different choices for E and provide examples of non-degenerate submanifolds which are not good.

4. The presentation ends with an appendix providing some elementary facts which are used throughout the thesis. The first section presents necessary local coordinate computations for Lorentzian manifolds with a parallel lightlike line. In particular, we construct suitable functions for Walker coordinates which are used in chapter two. The second section reviews the Killing form on $\mathfrak{so}(r, s)$ which is used in the third chapter.

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1 Holonomy, Foliations and Kähler Geometry

1.1 Pseudo-Riemannian Vector Bundles with Good Holonomy

In this presentation we assume all manifolds and vector bundles to be C^∞-differentiable and all manifolds to be connected second-countable Hausdorff spaces.

Definition 1.1.

1. *Let $\pi : (E, h) \to X$ be a real vector bundle over X where h is a C^∞-field of non-degenerate symmetric bilinear forms on the fibers of E with signature (r, s).*[1] *If ∇ is a metric connection on (E, h), i.e., $\nabla h = 0$ we say (E, h, ∇, π, X) is a pseudo-Riemannian vector bundle of signature (r, s) over X.*

2. *In case $E = TX$ we say (X, h) is a pseudo-Riemannian manifold of signature (r, s), where ∇ is supposed to be the Levi-Civita connection of (X, h).*

Given a local frame $(s_1, \dots, s_{r+s})$ of E over $U \subset X$ we may write $\nabla s_i = \sum_j s_j \omega_i^j$ where each ω_i^j is a 1-form over $U \subset X$. Let $\gamma : [a, b] \subset \mathbb{R} \to X$ be a piecewise smooth curve in X and V_t a C^∞-section of E along γ. We say V_t is parallel along γ if $\frac{\nabla V_t}{dt} = 0$, i.e., if locally

$$\dot{V}_t^i + \sum_j \omega_j^i(\dot{\gamma}(t)) V_t^j = 0 \qquad \forall i \in \{1, \dots, r+s\},$$

where $V_t =: \sum_i V^i s_i$. For any $v \in E_{\gamma(a)}$ we can find a parallel section V_t along γ with $V_a = v$ to which we refer as the *parallel displacement* of v along γ. Hence, we derive a linear isomorphism $\tau_\gamma^\nabla : E_{\gamma(a)} \to E_{\gamma(b)}$ such that $h_{\gamma(a)} = (\tau_\gamma^\nabla)^* h_{\gamma(b)}$.

Definition 1.2. *Let (E, h, ∇, π, X) be a pseudo-Riemannian vector bundle of signature (r, s) over X and $p \in X$. Define*

- $Hol_p(\nabla) := \{\tau_\gamma^\nabla : \gamma$ *-a closed piecewise C^∞-curve around p*$\} \subset O(E_p, h_p)$,
- $Hol_p^0(\nabla) := \{\tau_\gamma^\nabla \in Hol_p(\nabla) : 0 = [\gamma] \in \pi_1(X, p)\} \subset Hol_p(\nabla)$.

We call $Hol_p(\nabla) \hookrightarrow O(E_p, h_p)$ the holonomy representation of ∇ at p and $Hol_p(\nabla)$ the holonomy group of ∇ at p. Moreover, we refer to $Hol_p^0(\nabla) \hookrightarrow SO_0(E_p, h_p)$ as the restricted holonomy representation of ∇ at p and to $Hol_p^0(\nabla)$ as the restricted holonomy group of ∇ at p.

[1] In our convention r is the number of negative eigenvalues while s is the number of positive eigenvalues.

It can be proved that $Hol_p(\nabla) \hookrightarrow O(E_p, h_p)$ and $Hol_p^0(\nabla) \hookrightarrow SO_0(E_p, h_p)$ are representations of Lie subgroups [KN96]. Moreover, $Hol_p^0(\nabla)$ is the connected identity-component and a normal subgroup of $Hol_p(\nabla)$. In particular, there is a natural group epimorphism

$$\pi_1(X) \twoheadrightarrow Hol_p(\nabla)/Hol_p^0(\nabla)$$

and we have $\tau_\gamma^\nabla Hol_p(\nabla) {\tau_\gamma^\nabla}^{-1} = Hol_q(\nabla)$ where $\gamma : [a, b] \to M$ is a curve with $\gamma(a) = p$ and $\gamma(b) = q$. We write $\mathfrak{hol}_p(\nabla)$ for the Lie algebra of $Hol_p(\nabla)$ and $\mathfrak{hol}_p(\nabla) \hookrightarrow \mathfrak{so}(E_p, h_p)$ for its induced representation. In particular, $\mathfrak{hol}_p(\nabla) \hookrightarrow \mathfrak{so}(E_p, h_p)$ uniquely defines $Hol_p^0(\nabla) \hookrightarrow SO_0(E_p, h_p)$ and vice versa. Using an orthonormal basis for (E_p, h_p) we may identify $O(E_p, h_p)$ with $O(r, s)$ and $Hol(\nabla)$ with a subgroup $H_1 \subset O(r, s)$. By another choice of an orthonormal basis we will derive an identification of $Hol(\nabla)$ with a subgroup $H_2 = aH_1a^{-1} \subset O(r, s)$ for some $a \in O(r, s)$. Hence, we may think of *the* holonomy group of ∇ as a subgroup of $O(r, s)$ which is defined up to conjugation. Moreover, a similar statement holds for the restricted holonomy group as well as the holonomy algebra. Finally, if $F : \tilde{X} \to X$ is the universal covering of X and

$$\begin{array}{ccc} F^*E & \xrightarrow{\tilde{F}} & E \\ {\scriptstyle F^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \tilde{X} & \xrightarrow{F} & X \end{array}$$

the corresponding pullback of E then $(F^*E, F^*h, F^*\nabla, F^*\pi, \tilde{X})$ is a pseudo-Riemannian vector bundle over $\tilde{X}$ such that $Hol(F^*\nabla) = Hol^0(\nabla)$. For any open subsets $p \in V \subset U \subset X$ we may consider the restrictions of (E, h, ∇, π, X) to U and V. In particular, we derive $Hol_p^0(\nabla|_V) \subset Hol_p^0(\nabla|_U) \subset Hol_p^0(\nabla)$. This suggests the following

Definition 1.3. *Let (E, h, ∇, π, X) be a pseudo-Riemannian vector bundle of signature (r, s) over X and $p \in X$. For any sequence $(U_k)_{k\in\mathbb{N}}$ of connected open subsets of X such that $\{p\} = \bigcap_{k\in\mathbb{N}} U_k$ and $U_{k+1} \subset \bar{U}_k$ for all $k \in \mathbb{N}$ we define the local holonomy group at $p \in X$ by $Hol_p^{loc}(\nabla) := \bigcap_{k\in\mathbb{N}} Hol_p^0(U_k, \nabla|_{U_k})$ and local holonomy algebra $\mathfrak{hol}_p^{loc}(\nabla)$ as the Lie algebra of $Hol_p^{loc}(\nabla)$.*

There is no classification of holonomy groups of vector bundles with connection. E.g., any closed subgroup of $GL(m, \mathbb{R})$ can be realized as the holonomy group of a linear connection on a manifold [HO56]. Hence, we have to focus on holonomy groups with additional structures.

Definition 1.4. *Let $\mathfrak{g} \subset \mathfrak{so}(\mathcal{E}, h)$ for a pseudo-Euclidean vector space $(\mathcal{E}, h)$.*

1. *The space of algebraic curvature tensors with values in $\mathfrak{g}$ is given by*

$$\mathcal{K}(\mathfrak{g}) := \{R \in \Lambda^2\mathcal{E}^* \otimes \mathfrak{g} : R(x, y)z + R(y, z)x + R(z, x)y = 0\}.$$

2. *The space of algebraic weak curvature tensors with values in $\mathfrak{g}$ is given by*

$$\mathcal{B}_h(\mathfrak{g}) := \{Q \in \mathcal{E}^* \otimes \mathfrak{g} : h(Q(x)y, z) + h(Q(y)z, x) + h(Q(z)x, y) = 0\}.$$

Moreover, we say $\mathfrak{g}$ *is a Berger algebra if* $\mathfrak{g} = span\{R(x,y) : R \in \mathcal{K}(\mathfrak{g}),\ x,y \in \mathcal{E}\}$ *and a weak Berger algebra*[2] *if* $\mathfrak{g} = span\{Q(x) : Q \in \mathcal{B}_h(\mathfrak{g}),\ x \in \mathcal{E}\}$.

As we will see below, (weak) Berger algebras are the key tool to the study of holonomy representations of the tangent and the normal bundle of pseudo-Riemannian (sub-) manifolds. Therefore, we introduce the following

Definition 1.5. *A pseudo-Riemannian vector bundle* (E, h, ∇, π, X) *of signature* (r, s) *over* X *has*

- *good holonomy if* $\mathfrak{hol}_p(\nabla)$ *is a Berger algebra and*
- *almost good holonomy if* $\mathfrak{hol}_p(\nabla)$ *is a weak Berger algebra.*

Let $G \subset O(\mathcal{E}, h)$ be a subgroup for some pseudo-Euclidean vector space $(\mathcal{E}, h)$ and assume G does not leave any proper non-degenerate subspace invariant. In contrast to the Riemannian case G may leave a degenerate subspace $E \subset \mathcal{E}$ invariant. Thus, $E \cap E^\perp$ is an isotropic G-invariant subspace. In order to make the following statements simple we introduce

Definition 1.6. *Let* $(\mathcal{E}, h)$ *be a pseudo-Euclidean vector space. A subgroup* $G \subset O(\mathcal{E}, h)$ *acts weakly irreducibly with index* r *if*

- *there is no proper non-degenerate* G*-invariant subspace* $E \subset \mathcal{E}$*,*
- *there is a* G*-invariant isotropic subspace* $\Xi \subset \mathcal{E}$ *such that* $\dim \Xi = r$*,*
- *any* G*-invariant isotropic subspace* $\tilde{\Xi}$ *satisfies* $\dim \tilde{\Xi} \leq r$*.*

The same way we define weakly irreducible subalgebras $\mathfrak{g} \subset \mathfrak{so}(\mathcal{E}, h)$ *with index* r*.*

In the classification of holonomy representations of Riemannian manifolds it is crucial to reduce the problem to irreducible representations of Berger algebras. The reduction is possible since the representation of a reducible Berger algebra is completely reducible and moreover splits into a product of representations of Berger algebras. This property was shown in [BL52] and motivates

Definition 1.7. *Let* $\mathfrak{g} \subset \mathfrak{so}(\mathcal{E}, h)$ *for a pseudo-Euclidean vector space* $(\mathcal{E}, h)$*. We say* $\mathfrak{g}$ *has the Borel-Lichnérowicz property if there is an orthogonal decomposition* $\mathcal{E} = E_0 \oplus \ldots \oplus E_\ell$ *into non-degenerate* $\mathfrak{g}$*-invariant subspaces and a corresponding decomposition* $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_\ell$ *into commuting ideals such that each* $\mathfrak{g}_j \subset \mathfrak{so}(E_j, h|_{E_j})$ *acts weakly irreducibly on* E_j *and trivially on* E_i *for* $i \neq j$*.*

Now, we have

[2] Weak Berger algebras have been introduced by Leistner in [Lei07]. For any $R \in \mathcal{K}(\mathfrak{g})$ and fixed $t \in \mathcal{E}$ we have $\langle R(t,x)y, z\rangle + \langle R(t,y)z, x\rangle + \langle R(t,z)x, y\rangle = 0$, i.e., $Q(\cdot) := R(t, \cdot) \in \mathcal{B}_h(\mathfrak{g})$. Hence, any Berger algebra $\mathfrak{g} \subset \mathfrak{so}(\mathcal{E}, h)$ is a weak Berger algebra and the name is justified.

Proposition 1.8. *Let $(\mathcal{E}, h)$ be a pseudo-Euclidean vector space and let $\mathfrak{g} \subset \mathfrak{so}(\mathcal{E}, h)$ be a (weak) Berger algebra. Then $\mathfrak{g}$ has the Borel-Lichnérowicz property. Moreover, each ideal $\mathfrak{h}_j \subset \mathfrak{so}(E_j, h|_{E_j})$ is a (weak) Berger algebra.*

Proof. The proof of this fact is classic and can be found in [Sim62] and [Lei07]. Since it is very short and crucial to this exposition we reproduce it here.

First, we derive the decomposition $\mathcal{E} = E_0 \oplus \ldots \oplus E_\ell$, where E_0 is a maximal non-degenerate space on which $\mathfrak{g}$ acts trivially since $\mathfrak{g} \subset \mathfrak{so}(\mathcal{E}, h)$. We define $\mathfrak{g}_j := \mathrm{span}\{R(x, y) : x, y \in E_j,\ R \in \mathcal{K}(\mathfrak{g})\}$ if $\mathfrak{g}$ is a Berger algebra and $\mathfrak{g}_j := \mathrm{span}\{Q(x) : x \in E_j,\ Q \in \mathcal{B}_h(\mathfrak{g})\}$ otherwise. Since E_j is invariant we conclude that $\mathfrak{g}_j \subset \mathfrak{so}(E_j, h|_{E_j})$ is a (weak) Berger algebra. Let $i \neq j$ and $x \in E_i,\ y \in E_j$. Then $0 = \langle R(\cdot, \cdot)x, y\rangle = \langle R(x, y)\cdot, \cdot\rangle$, i.e., $R(x, y) = 0$. Moreover, for $x \in E_0$ and $y \in \mathcal{E}$ we have $0 = \langle R(\cdot, \cdot)x, y\rangle = \langle R(x, y)\cdot, \cdot\rangle$ and $0 = \langle Q(x)y, \cdot\rangle = \langle Q(y)x, \cdot\rangle - \langle Q(\cdot)x, y\rangle$, i.e., $R(x, \cdot) = Q(x) = 0$. Hence, linearity implies $\mathfrak{g} = \sum_j \mathfrak{g}_j$. For $x, y \in E_j,\ z \in E_i$ the Bianchi identity implies $R(x, y)z = -R(y, z)x - R(z, x)y = 0$ and $\langle Q(z)x, y\rangle = -\langle Q(x)y, z\rangle - \langle Q(y)z, x\rangle = 0$. Therefore, $\mathfrak{g}_i \cap \mathfrak{g}_j = 0$ for $i \neq j$. Finally, for $x, y \in E_i,\ z, t \in E_j$ invariance implies $R(x, y) \circ R(z, t) = 0$ and $Q(x) \circ Q(z) = 0$, i.e., the $\mathfrak{g}_j$ are mutually commuting ideals. □

Finally, we summarize the known classification results for (weak) Berger algebras. For the proofs we refer to [Bry00] and [Lei07].

Theorem 1.9 (Berger et al., Leistner)**.**

1. *Let $\mathfrak{g} \subset \mathfrak{so}(\mathcal{E}, h)$ be an irreducible Berger algebra where $(\mathcal{E}, h)$ is a pseudo-Euclidean vector space. If $\mathfrak{g} \neq \mathfrak{so}(\mathcal{E}, h)$ then $\mathfrak{g}$ is the holonomy representation of an irreducible pseudo-Riemannian symmetric space or given by the following list*

$$\begin{aligned}
\mathfrak{u}(r, s), \mathfrak{su}(r, s) &\subset \mathfrak{so}(2r, 2s),\\
\mathfrak{sp}(1) \oplus \mathfrak{sp}(r, s), \mathfrak{sp}(r, s) &\subset \mathfrak{so}(4r, 4s),\\
\mathfrak{so}(r, \mathbb{C}) &\subset \mathfrak{so}(r, r),\\
\mathfrak{sp}(r, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) &\subset \mathfrak{so}(2r, 2r),\\
\mathfrak{sp}(r, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) &\subset \mathfrak{so}(4r, 4r),\\
\mathfrak{g}_2 &\subset \mathfrak{so}(7),\\
\mathfrak{g}_2^{\mathbb{C}} \subset \mathfrak{so}(7, \mathbb{C}) &\subset \mathfrak{so}(7, 7),\\
\mathfrak{g}_2^2 &\subset \mathfrak{so}(4, 3),\\
\mathfrak{spin}(7) &\subset \mathfrak{so}(8),\\
\mathfrak{spin}(7, \mathbb{C}) \subset \mathfrak{so}(8, \mathbb{C}) &\subset \mathfrak{so}(8, 8),\\
\mathfrak{spin}(4, 3) &\subset \mathfrak{so}(4, 4).
\end{aligned}$$

2. *If $\mathfrak{g} \subset \mathfrak{so}(n)$ is an irreducible weak Berger algebra then it is a Berger algebra.*

□

Remark 1.10.

1. In [DSO01] Di Scala and Olmos have shown that $\mathfrak{so}(1,n)$ is the only irreducible subalgebra of $\mathfrak{so}(1,n)$. Hence, the Lorentzian case of Theorem 1.9 can be deduced from this result. In particular, any irreducible Lorentzian weak Berger algebra is given by $\mathfrak{so}(1,n)$.

2. In [DSL08] Di Scala and Leistner have shown that any irreducible subalgebra of $\mathfrak{so}(2,n)$ is given by the subalgebras in Theorem 1.9 and $i\mathbb{R}\oplus\mathfrak{so}(2,n)\subset\mathfrak{u}(1,n)$. In particular, any irreducible weak Berger algebra in $\mathfrak{so}(2,n)$ is of that form. □

The holonomy principle relates holonomy invariant tensors in a fiber to parallel sections in the corresponding tensor bundle. More precisely, we have the following

Theorem 1.11 (Holonomy Principle)**.** *Let (E,∇) be a vector bundle over X with connection ∇. Then we have three sets of equivalent conditions:*[3]

1.
- *There is a ∇-parallel subbundle $F\subset E$, i.e., $\nabla_{V_p}\xi\in F_p$ for any section $\xi\in\Gamma(F)$ and all $V_p\in T_pX$.*
- *There is a subbundle $F\subset E$ such that for any curve $\gamma:[a,b]\to X$ and any $v\in F_{\gamma(a)}$ we have $\tau_\gamma^\nabla(v)\in F_{\gamma(b)}$ for the parallel displacement of v along γ.*
- *There exists $p\in X$ and a subspace $F_p\subset E_p$ such that $Hol_p(\nabla)\cdot F_p\subset F_p$.*

2.
- *There is a ∇-parallel section $\xi\in\Gamma(E)$.*
- *There is a section $\xi\in\Gamma(E)$ such that for any curve $\gamma:[a,b]\to X$ we have $\tau_\gamma^\nabla(\xi_{\gamma(a)})=\xi_{\gamma(b)}$.*
- *There exists $p\in X$ and a vector $\xi\in E_p$ such that $Hol_p\cdot\xi_p=\xi_p$.*

3.
- *There is a splitting $E=F\oplus\tilde{F}$ into ∇-parallel subbundles F and $\tilde{F}$.*
- *For some $p\in X$ there is a splitting $E_p=F_p\oplus\tilde{F}_p$ such that $Hol_p(\nabla)\cdot F_p\subset F_p$ and $Hol_p(\nabla)\cdot\tilde{F}_p\subset\tilde{F}_p$.*

Moreover, if E is a pseudo-Riemannian vector bundle we may substitute orthogonal splitting for splitting.

□

We may consider the pullback of E to the universal cover of X and derive a corresponding holonomy principle for the restricted holonomy group and the holonomy algebra. In the previous theorem we lose the equivalences if we replace $Hol_p(\nabla)$ by $\mathfrak{hol}_p(\nabla)$. Nevertheless, each holonomy condition implies the others locally. In particular, we derive a local version once we substitute $\mathfrak{hol}_p^{loc}(\nabla)$ for $Hol_p(\nabla)$ and add *locally around p* to the other conditions. Using Proposition 1.8 as well as Theorem 1.9 and the holonomy principle we derive

[3] The parallel displacement and the holonomy representation of a vector bundle with arbitrary connection is defined in the same way as for pseudo-Riemannian vector bundles.

Corollary 1.12 (Algebraic de Rham Decomposition)**.** *For a pseudo-Riemannian vector bundle (E, h, ∇, π, X) with (almost) good holonomy over a simply connected base X we have an orthogonal decomposition*

$$E = E_0 \oplus \ldots \oplus E_\ell$$

into $\mathfrak{hol}(\nabla)$-invariant (non-degenerate) pseudo-Riemannian subbundles with (almost) good holonomy and a corresponding decomposition

$$\mathfrak{hol}(\nabla) = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_\ell$$

into ideals such that for all $j \geq 1$

- $\mathfrak{h}_j \subset \mathfrak{so}(E_j, h|_{E_j})$ *acts weakly irreducibly on E_j and trivially on E_i for $i \neq j$,*
- $\mathfrak{h}_j = \mathfrak{hol}(E_j, \nabla|_{E_j})$*,*
- $\mathfrak{h}_j$ *acts as one of the algebras given in Theorem 1.9 and Remark 1.10 unless $\mathfrak{h}_j$ is not irreducible.*

□

In the previous theorem we did not mention how $\mathfrak{h}_j$ acts if it is weakly irreducible with index $r > 0$. In fact, Galaev proved in [Gal04] that there is no classification of weakly irreducible (weak) Berger algebras. However, the situation is different if we restrict to Lorentzian signature or the holonomy of the normal bundle of certain submanifolds in pseudo-Euclidean space forms.

For a pseudo-Riemannian vector bundle (E, h, ∇, π, X) let $R^\nabla_p(V, W) := [\nabla_V, \nabla_W] - \nabla_{[V,W]}$ be the curvature tensor of ∇ at $p \in X$. If $\gamma : [0, 1] \to X$ is a piecewise smooth curve in X such that $\gamma(0) = p$ and if τ^∇_γ is the parallel displacement w.r.t. ∇ along γ we define

$$R_p^{\tau_\gamma^\nabla}(v, w) := \tau_\gamma^{\nabla -1} \circ R^\nabla_{\gamma(1)}(v, w) \circ \tau^\nabla_\gamma$$

for $v, w \in T_{\gamma(1)}X$.

Theorem 1.13 (Ambrose-Singer Holonomy Theorem [KN96])**.** *If (E, h, ∇, π, X) is a pseudo-Riemannian vector bundle and $p \in X$ then*

$$\mathfrak{hol}_p(\nabla) = span\{R_p^{\tau_\gamma^\nabla}(\tilde{\tau}_\gamma(v), \tilde{\tau}_\gamma(w)) : v, w \in T_pX,\ \gamma : [0, 1] \to X,\ \gamma(0) = p\},$$

where $\tilde{\tau}_\gamma$ is the parallel displacement w.r.t. some connection $\tilde{\nabla}$ on TX along γ. □

If the vector bundle appears to be the tangent bundle TX of a pseudo-Riemannian manifold X the Ambrose-Singer holonomy theorem implies that TX has good holonomy. Moreover, we have $[V_1, V_2] = \nabla_{V_1}V_2 - \nabla_{V_2}V_1$ as the Levi-Civita connection is torsion-free. Hence, any invariant subbundle $E \subset TX$ is involutive and by Frobenius' theorem locally the tangent bundle of a maximal integral manifold. By a more detailed analysis de Rham [dR52] and Wu [Wu64], [Wu67] proved the following

Theorem 1.14 (Geometric de Rham Decomposition)**.** *Suppose (X, h) is a pseudo-Riemannian manifold. Moreover, let $T_pX = E_0 \oplus \ldots \oplus E_\ell$ be the orthogonal decomposition into non-degenerate $\mathfrak{hol}(X, h)$-invariant subspaces and $\mathfrak{hol}(X, h) = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_\ell$ the corresponding decomposition such that each $\mathfrak{h}_j$ acts weakly irreducibly on E_j and trivially on E_i for $i \neq j$.*

1. *For $0 \leq j \leq \ell$ there are immersed submanifolds (X_j, h_j) with $p \in X_j$ and $T_pX_j = E_j$ such that there is a neighborhood of p in X which is isometric to a product $U_0 \times \ldots \times U_\ell$ where each U_j is an open neighborhood of X_j with its induced metric.*

2. *The isometry is global if (X, h) is simply-connected and complete, i.e., $(X, h) = (\mathbb{R}^{\dim E_0}, \langle\cdot,\cdot\rangle) \times (X_1, h_1) \times \ldots \times (X_\ell, h_\ell)$. In particular, $\mathfrak{h}_j = \mathfrak{hol}(X_j, h_j)$. The decomposition is unique up to order if the maximal subspace on which $\mathfrak{hol}(X, h)$ acts trivially is non-degenerate.*

3. *Any irreducible $\mathfrak{h}_j$ is given by Theorem 1.9.*

□

1.2 Geometry of Kähler Manifolds

In this section we briefly introduce sheaf cohomology and explain the concepts of complex geometry used throughout the presentation. For a comprehensive introduction we refer to [Huy05] and [Voi07]. Let X be a topological space. We define the category $\mathcal{U}_X$ as follows. Each open subset of X is an object of $\mathcal{U}_X$ and if U, V are open subsets of X then

$$Hom_{\mathcal{U}_X}(U,V) := \begin{cases} \{U \hookrightarrow V\} & \text{if } U \subset V, \\ \emptyset & \text{otherwise.} \end{cases}$$

If $\mathcal{C}$ is any category then a presheaf $\mathcal{F}$ on X with values in $\mathcal{C}$ is a contravariant functor $\mathcal{F} : \mathcal{U}_X \to \mathcal{C}$. If $\mathcal{C}$ has a zero object 0 we require $\mathcal{F}(\emptyset) = 0$. If $\mathcal{F}, \mathcal{G}$ are presheaves with values in $\mathcal{C}$ on X then a natural transformation $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism from $\mathcal{F}$ to $\mathcal{G}$, i.e., we have a $\mathcal{C}$-morphism $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for any open subset $U \subset X$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{F(f)} & \mathcal{F}(V) \\ {\scriptstyle \varphi_U} \downarrow & & \downarrow {\scriptstyle \varphi_V} \\ \mathcal{G}(U) & \xrightarrow{G(f)} & \mathcal{G}(V) \end{array}$$

for each $f \in Hom_{\mathcal{U}_X}(U,V)$. Let $\mathcal{F}$ be a presheaf of abelian groups (or R-modules, vector spaces, rings, ...). If $s \in \mathcal{F}(U)$ and $V \subset U$ we write $s|_V := \mathcal{F}(V \hookrightarrow U)(s)$. Moreover, we say $\mathcal{F}$ is a sheaf on X if the following holds for any open subset $U \subset X$ and any open covering $U = \bigcup_{i \in I} U_i$:

- If $s, t \in \mathcal{F}(U)$ and $s|_{U_i} = t|_{U_i}$ for all $i \in I$ then $s = t$.

- If we have $s_i \in \mathcal{F}(U_i)$ for all $i \in I$ and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ then there exists $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

A morphism of sheaves on X is a morphism of its underlying presheaves. For any (pre-)sheaf $\mathcal{F}$ on X and any $x \in X$ the stalk of $\mathcal{F}$ at x is given by

$$\mathcal{F}_x := \{(U,s) : x \in U \subset X,\ s \in \mathcal{F}(U)\}/_\sim,$$

where $(U_1, s_1) \sim (U_2, s_2)$ if there exists $x \in V \subset U_1 \cap U_2$ such that $s_1|_V = s_2|_V$. A morphism of (pre-)sheaves on X induces a morphism on the stalks. If $\mathcal{F}^1, \mathcal{F}^2, \mathcal{F}^3$ are sheaves of abelian groups[4] then we say the sequence

$$0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \mathcal{F}^3 \to 0$$

[4] More generally, we may require $\mathcal{F}^i$ to be sheaves with values in an abelian category.

is exact if and only if for each $x \in X$ the induced sequence on the stalks is exact. If X is a smooth manifold and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ then we define the sheaves $\mathcal{C}^\infty_\mathbb{K}, \mathcal{C}^*_\mathbb{K}, \underline{\mathbb{K}}, \underline{\mathbb{Z}}, \underline{\mathbb{Q}}$ by

$$\begin{aligned} \mathcal{C}^\infty_\mathbb{K}(U) &:= \{f : U \to \mathbb{K} : \ f \text{ is smooth}\}, \\ \mathcal{C}^*_\mathbb{K}(U) &:= \text{multiplicative group of nowhere zero } f \in \mathcal{C}^\infty_\mathbb{K}(U), \\ \underline{\mathbb{K}}(U),\ \underline{\mathbb{Z}}(U),\ \underline{\mathbb{Q}}(U) &:= \text{locally constant functions with values in } \mathbb{K}, \mathbb{Z}, \mathbb{Q}. \end{aligned}$$

If X is a complex manifold the associated almost complex structure J induces a splitting $T_xX \otimes \mathbb{C} = T^{1,0}_x X \oplus T^{0,1}_x X$. This way we define vector bundles $\Lambda^{p,q}X := \Lambda^p(T^{1,0}X)^* \otimes_\mathbb{C} \Lambda^q(T^{0,1}X)^*$ and $\Lambda^k_\mathbb{K} X = \Lambda^k(TX \otimes \mathbb{K})^*$. We define the sheaves $\mathcal{A}^k_{X,\mathbb{K}}, \mathcal{A}^{p,q}_X, \mathcal{O}_X, \mathcal{O}^*_X$ by

$$\begin{aligned} \mathcal{A}^k_{X,\mathbb{K}}(U) &:= \{s : U \to \Lambda^k_\mathbb{K} X : \ s \text{ is a } C^\infty\text{-section}\}, \\ \mathcal{A}^{p,q}_X(U) &:= \{s : U \to \Lambda^{p,q} X : \ s \text{ is a } C^\infty\text{-section}\}, \\ \mathcal{O}_X(U) &:= \{f : U \to \mathbb{C} : \ f \text{ is holomorphic}\}, \\ \mathcal{O}^*_X(U) &:= \text{multiplicative group of nowhere zero } f \in \mathcal{O}_X(U). \end{aligned}$$

Given an open covering of X by holomorphic coordinate charts $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{C}^{\dim_\mathbb{C} X}$ define $\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ and for $z \in U_i \cap U_j$ the matrix $J_{ij}(\varphi_j(z)) := (\frac{\partial \varphi^k_{ij}}{\partial z^\ell}(\varphi_j(z)))_{k,\ell}$. The holomorphic tangent bundle $\mathcal{T}_X$ is the holomorphic rank $\dim_\mathbb{C} X$ vector bundle on X whose transition cocycle is given by $J_{ij}(\varphi_j(z))$. The holomorphic cotangent bundle Ω_X is the dual bundle of $\mathcal{T}_X$. The sheaf of holomorphic p-forms Ω^p_X is the sheaf of holomorphic sections of $\Lambda^p \Omega_X$.

Since $\mathcal{A}^k_{X,\mathbb{C}} = \bigoplus_{k=p+q} \mathcal{A}^{p,q}_X$ we can define $\partial := pr_{p+1,q} \circ d : \mathcal{A}^{p,q}_X \to \mathcal{A}^{p+1,q}_X$ and $\bar\partial := pr_{p,q+1} \circ d : \mathcal{A}^{p,q}_X \to \mathcal{A}^{p,q+1}_X$. Moreover, if X is a complex manifold we conclude $d = \partial + \bar\partial$. On the complex manifold X we define the (p,q)-Dolbeault cohomology by

$$H^{p,q}(X) = \frac{\operatorname{Ker}(\bar\partial : \mathcal{A}^{p,q}_X(X) \to \mathcal{A}^{p,q+1}_X(X))}{\operatorname{Im}(\bar\partial : \mathcal{A}^{p,q-1}_X(X) \to \mathcal{A}^{p,q}_X(X))}.$$

Let I be a countable ordered set and $\mathcal{U} := \{U_i : i \in I\}$ a locally finite open covering of the topological space X. We write $U_{i_0 \dots i_p} := U_{i_0} \cap \dots \cap U_{i_p}$. If $\mathcal{F}$ is a sheaf of abelian groups on X let $\Gamma(U_{i_0 \dots i_p}, \mathcal{F}) := \mathcal{F}(U_{i_0 \dots i_p})$ and define

$$\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \dots < i_p} \Gamma(U_{i_0 \dots i_p}, \mathcal{F}).$$

as well as $\delta_p : \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^{p+1}(\mathcal{U}, \mathcal{F})$, $\sigma = \prod \sigma_{i_0 \dots i_p} \mapsto \delta_p(\sigma)$ where $(\delta_p \sigma)_{i_0 \dots i_{p+1}} := \sum_{k=0}^{p+1} (-1)^k \sigma_{i_0 \dots \widehat{i_k} \dots i_{p+1}}|_{U_{i_0 \dots i_{p+1}}}$. Since $\delta_{p+1} \circ \delta_p = 0$ we derive a complex $\check{\mathcal{C}}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_0} \check{\mathcal{C}}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_1} \check{\mathcal{C}}^2(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_2} \dots$

Definition 1.15. *The p-th Čech cohomology group of $\mathcal{F}$ w.r.t. $\mathcal{U}$ is given by*

$$\check{H}^p(\mathcal{U},\mathcal{F}) = \frac{Ker(\delta_p : \check{\mathcal{C}}^p(\mathcal{U},\mathcal{F}) \to \check{\mathcal{C}}^{p+1}(\mathcal{U},\mathcal{F}))}{Im(\delta_{p-1} : \check{\mathcal{C}}^{p-1}(\mathcal{U},\mathcal{F}) \to \check{\mathcal{C}}^p(\mathcal{U},\mathcal{F}))}.$$

If $\mathcal{V}$ is a locally finite open covering of X which is a refinement of $\mathcal{U}$ then there exists a natural group homomorphism $\check{H}^p(\mathcal{U},\mathcal{F}) \to \check{H}^p(\mathcal{V},\mathcal{F})$. Thus, by passing to the direct limit we can define the p-th Čech cohomology of $\mathcal{F}$ to be $\check{H}^p(X,\mathcal{F}) := \varinjlim \check{H}^p(\mathcal{U},\mathcal{F})$. The definition implies $\check{H}^0(\mathcal{U},\mathcal{F}) = \Gamma(X,\mathcal{F})$ and a combinatorial way of computing cohomology is given by

Theorem 1.16 (Leray, [Voi07, Thm. 4.41])**.** *Let $\mathcal{F}$ be a sheaf of abelian groups on the topological manifold X and $\mathcal{U}$ as above. If $\check{H}^p(U_{i_0\ldots i_q},\mathcal{F}) = 0$ for all $p > 0$ and all $i_0 \ldots i_q$ then $\check{H}^p(X,\mathcal{F}) = \check{H}^p(\mathcal{U},\mathcal{F})$ for all $p \geq 0$.*[5] □

Let $\mathcal{F}$ be a sheaf of abelian groups on the topological manifold X. We say $\mathcal{F}$ is a fine sheaf if any open cover $\{U_i\}_{i\in I}$ of X admits a partition of unity of $\mathcal{F}$, i.e., a family of endomorphisms $(f_i)_{i\in I}$ of $\mathcal{F}$ such that $\{i \in I : f_i|_p \neq 0\}$ is finite for any $p \in X$ and $\sum_{i\in I} f_i = \mathrm{id}_{\mathcal{F}}$ as well as $\mathrm{supp} f_i \subset U_i$ $\forall i \in I$. In general, we have $H^p(X,\mathcal{F}) = 0$ for all $p > 0$ if $\mathcal{F}$ is a fine sheaf and examples of fine sheaves are given by $\mathcal{C}^\infty_{\mathbb{K}}, \mathcal{A}^k_{X,\mathbb{K}}, \mathcal{A}^{p,q}_X$. We have

Theorem 1.17 ([Voi07, Ch. 4])**.**

1. *If X is a topological manifold then $H^p_{sing}(X,A) = \check{H}^p(X,\underline{A})$ for all $p \geq 0$ where $A \in \{\mathbb{Z},\mathbb{Q},\mathbb{K}\}$ and $H^p_{sing}(X,A)$ is the singular cohomology with values in A.*

2. *If X is a smooth manifold then $H^p_{dR}(X,\mathbb{K}) = \check{H}^p(X,\underline{\mathbb{K}})$ for all $p \geq 0$ where $H^p_{dR}(X,\mathbb{K})$ is the de Rham cohomology with values in $\mathbb{K}$.*

3. *If X is a smooth manifold then there is a bijection between $H^1(X,\mathcal{C}^*_{\mathbb{C}})$ and the set of isomorphism classes of complex C^∞-vector bundles on X.*

4. *If X is a complex manifold then $H^{p,q}(X) = \check{H}^q(X,\Omega^p)$.*

5. *If X is a complex manifold then there is a bijection between $H^1(X,\mathcal{O}^*_X)$ and the set of isomorphism classes of holomorphic line bundles on X.* □

In fact, the tensor product induces the structure of an abelian group on the set of isomorphism classes of holomorphic line bundles making the above bijection an isomorphism. We refer to this group as the Picard group $Pic(X)$ of X.

[5] If $\check{H}^p(U_{i_0\ldots i_q},\mathcal{F})$ is replaced by $H^p(U_{i_0\ldots i_q},\mathcal{F})$ then we may replace $\check{H}^p(X,\mathcal{F})$ by $H^p(X,\mathcal{F})$ where $H^p(\cdot,\mathcal{F})$ denotes the derived functor cohomology which we do not introduce here. In this case, we may drop the condition on X to be a manifold. In the following, if $\mathcal{F}$ is a sheaf of abelian groups on the topological manifold X then we write $H^p(X,\mathcal{F})$ for its Čech/derived functor cohomology.

Any exact sequence $0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \mathcal{F}^3 \to 0$ of sheaves of abelian groups on the topological manifold X induces a long exact sequence

$$0 \to \Gamma(X, \mathcal{F}^1) \to \Gamma(X, \mathcal{F}^2) \to \Gamma(X, \mathcal{F}^3) \xrightarrow{\delta} H^1(X, \mathcal{F}^1) \to \dots$$

For any smooth manifold X there exists an exact sequence of sheaves $0 \to \underline{\mathbb{Z}} \xrightarrow{\iota} \mathcal{C}^\infty_{\mathbb{C}} \xrightarrow{\exp} \mathcal{C}^*_{\mathbb{C}} \to 0$ where ι is the natural inclusion and exp is given by $f \mapsto e^{2\pi\sqrt{-1}f}$. Since $\mathcal{C}^\infty_{\mathbb{C}}$ is a fine sheaf the long exact sequence implies $H^1(X, \mathcal{C}^*_{\mathbb{C}}) \overset{\delta}{\cong} H^2(X, \mathbb{Z})$. On a complex manifold X the exponential sequence is the exact sequence $0 \to \underline{\mathbb{Z}} \xrightarrow{\iota} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}^*_X \to 0$. We derive a long exact sequence

$$\Gamma(X, \mathcal{O}^*) \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to Pic(X) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}).$$

The natural inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ induces a morphism $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$. We refer to the image $NS(X)$ of $Pic(X) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$ as the (real) *Neron-Severi group*. If X is compact then $\rho(X) := rk(NS(X))$ is the *Picard number* of X.

Let (M, J, g) be a Hermitian manifold, i.e., (M, g) is a Riemannian manifold and J is an integrable almost complex structure on M such that $g(J(\cdot), J(\cdot)) = g(\cdot, \cdot)$. We say $(X = (M, J), g)$ is Kähler if the Kähler form $\omega := g(J(\cdot), \cdot)$ is closed. The Kähler form is a real $(1,1)$-form and the class $[\omega] \in H^{1,1}(X)$ is the Kähler class of (X, g). On a complex manifold X let $A \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$. Using the natural inclusion $A \subset \mathbb{C}$ we define

$$H^{1,1}(X, A) := \mathrm{Im}(H^2(X, A) \to H^2(X, \mathbb{C})) \cap H^{1,1}(X)$$

Proposition 1.18 (Lefschetz Thm. on $(1,1)$-classes)**.** *If X is a compact Kähler manifold then $Im(Pic(X) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})) = H^{1,1}(X, \mathbb{Z})$, i.e., under the natural inclusion $NS(X) = H^{1,1}(X, \mathbb{Z})$.* □

On a Hermitian manifold $(X = (M, J), g)$ we have a natural orientation inducing a Hodge $*$-operator $\Lambda^k X \to \Lambda^{\dim_{\mathbb{R}} M - k} X$. Define

$$\begin{aligned} &L : \Lambda^k X \to \Lambda^{k+2} X, \quad \varphi \mapsto \omega \wedge \varphi && \text{(Lefschetz operator)},\\ &\Lambda : \Lambda^k X \to \Lambda^{k-2} X, \quad \varphi \mapsto (*^{-1} \circ L \circ *)(\varphi) && \text{(dual Lefschetz operator)}. \end{aligned}$$

Given an orthonormal basis $(e_1, Je_1, \dots, e_n, Je_n)$ and a real 2-form ψ we have $\Lambda\psi = \sum_{i=1}^n \psi(e_i, Je_i)$. The $\mathbb{C}$-linear extensions of $*, L, \Lambda$ are again denoted by $*, L, \Lambda$. Finally, we define $\partial^* := - * \circ \bar{\partial} \circ *$ and $\bar{\partial}^* := - * \circ \partial \circ *$ as well as $\Delta_\partial := \partial^*\partial + \partial\partial^*, \Delta_{\bar{\partial}} := \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^* : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q}(X)$.

Proposition 1.19 (Kähler identities)**.** *If (X, g) is a Kähler manifold then*

- $[\bar{\partial}, L] = [\partial, L] = [\bar{\partial}^*, \Lambda] = [\partial^*, \Lambda] = 0$ *and* $\Delta_\partial = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta_d$
- $[\bar{\partial}^*, L] = \sqrt{-1}\partial$, $[\partial^*, L] = -\sqrt{-1}\bar{\partial}$, $[\Lambda, \bar{\partial}] = -\sqrt{-1}\partial^*$, $[\Lambda, \partial] = \sqrt{-1}\bar{\partial}^*$
- $\Delta_{\bar{\partial}}$ *commutes with* $*, \partial, \bar{\partial}, \partial^*, \bar{\partial}^*, L, \Lambda$. □

If (X,g) is compact Kähler then we write $\mathcal{H}^{p,q}_{\bar\partial}(X,g)$ resp. $\mathcal{H}^k(X,g)$ for the space of $\Delta_{\bar\partial}$-harmonic (p,q)-forms resp. Δ_d-harmonic k-forms. By the Hodge theorem any class $[\alpha] \in H^{p,q}(X)$ has a unique harmonic representative $\alpha \in \mathcal{H}^{p,q}(X,g)$. In particular, there is a natural isomorphism $\mathcal{H}^{p,q}(X,g) \to H^{p,q}(X)$. By the Kähler identities, the operators L and Λ commute with the Laplacian $\Delta_{\bar\partial}$. Thus, there are induced maps $L : \mathcal{H}^{p,q}(X,g) \to \mathcal{H}^{p+1,q+1}(X,g)$ and $\Lambda : \mathcal{H}^{p,q}(X,g) \to \mathcal{H}^{p-1,q-1}(X,g)$.

Suppose (X,g) is a compact Kähler manifold. By the Hodge theorem we derive $L : H^{p,q}(X) \to H^{p+1,q+1}(X)$ and $\Lambda : H^{p,q}(X) \to H^{p-1,q-1}(X)$ which, in fact, depend on the Kähler class (and the complex structure) but not on the particular Kähler metric.

Definition 1.20. *Let $[\omega] \in H^{1,1}(X,\mathbb{R})$ be a Kähler class on the compact complex manifold X and $A \in \{\mathbb{Z},\mathbb{Q},\mathbb{R}\}$. The primitive cohomology of $(X,[\omega])$ is given by $H^{p,q}_{prim}(X) :=$ Ker$(\Lambda : H^{p,q}(X) \to H^{p-1,q-1}(X))$ and we define $H^{p,q}_{prim}(X,A) :=$ Im$(H^{p+q}(X,A) \to H^{p+q}(X,\mathbb{C})) \cap H^{p,q}_{prim}(X)$.*

Definition 1.21. *Let (X,g) be a compact Kähler manifold. A holomorphic symplectic structure on (X,g) is a closed $(2,0)$-form σ which is everywhere non-degenerate. Moreover, we say (X,g,σ) is a simple holomorphic symplectic manifold if X is simply connected and $H^{2,0}(X) = \mathbb{C}[\sigma]$.*

Proposition 1.22 (Beauville [Bea83, Prop. 4])**.** *Let $(X,\tilde g)$ be a compact Kähler manifold and $[\omega]$ any Kähler class on X. Then:*

1. *$X = (M,J)$ admits a holomorphic symplectic structure if and only if there are complex structures $J_1 = J, J_2, J_3 = J_1J_2$ and a unique Riemannian metric g which is Kähler w.r.t. all (M,J_i) such that $[g(J(\cdot),\cdot)] = [\omega]$. In particular, $Hol(M,g) \subset Sp(\frac{\dim_{\mathbb{C}} X}{2})$.*[6]
2. *$X = (M,J)$ admits a simple holomorphic symplectic structure if and only if there is $(J_1 = J, J_2, J_3, g)$ as above and $Hol(M,g) = Sp(\frac{\dim_{\mathbb{C}} X}{2})$.* □

Theorem 1.23 (Oguiso [Ogu00],[Ogu03])**.** *If (X,g,σ) is a simple holomorphic symplectic manifold then for all $0 \le k \le b_2(X) - 2$ there exists a simple holomorphic symplectic manifold X' such that X and X' are deformation equivalent and $\rho(X') = k$. In particular, X and X' are diffeomorphic.* □

Definition 1.24. *A Riemannian manifold (M,g) such that $\dim_{\mathbb{R}} M = 4n$ and $n \ge 2$ is quaternionic Kähler if $Hol(M,g) \subset Sp(n)Sp(1)$.*

Proposition 1.25 ([Bes87, 14.36])**.** *(M,g) is quaternionic Kähler if and only if there is a real rank 3 subbundle $E \subset End(TM)$ such that around each $x \in M$ there are local sections J_1, J_2, J_3 spanning E such that*

- *$J_i^2 = -id_{TM}$, $J_1J_2 = J_3$ and $g(J_i(\cdot), J_i(\cdot)) = g(\cdot,\cdot)$*
- *$\nabla^g_\cdot J_i \in \Gamma(U \subset M, E)$.* □

[6] We refer to (J_1, J_2, J_3, g) as a hyperkähler structure on M.

Given the natural metric on E we have the unit-sphere bundle $\pi : \mathcal{Z}_M \to M$ and call $\mathcal{Z}_M$ the *twistor space* of (M, g). By a theorem of Salamon [Bes87, 14.68] there exists an integrable almost complex structure J on $\mathcal{Z}_M$.

Theorem 1.26 (LeBrun & Salamon [BG08, Thm. 12.3.5])**.** *A compact quaternionic Kähler manifold (M, g) with positive Ricci curvature is simply connected. Moreover, $\pi_2(M)$ is finite unless (M, g) is isometric to the complex Grassmannian $Gr_2(\mathbb{C}^{n+2})$ with its quaternionic Kähler metric in which case $\pi_2(Gr_2(\mathbb{C}^{n+2})) = \mathbb{Z}$.* □

1.3 A Brief Review of Riemannian Foliations

On the smooth manifold X define $n := \dim_{\mathbb{R}} X$ and let $q \in \mathbb{N}$. Suppose $\mathfrak{A}$ is a C^∞-atlas on X with the following property: If $(U,\varphi),(V,\psi) \in \mathfrak{A}$ then $\rho := \psi \circ \varphi^{-1}$ has the form $\rho : \mathbb{R}^{n-q} \times \mathbb{R}^q \to \mathbb{R}^{n-q} \times \mathbb{R}^q$ such that $(x,y) \mapsto (\rho_1(x,y), \rho_2(y))$. In this case, we say $\mathfrak{A}$ is a q-codimensional foliated atlas and two such atlases are equivalent if their union is a q-codimensional foliated atlas. A *foliation* $\mathcal{F}$ of dimension $n-q$ on X is an equivalence class of q-codimensional foliated atlases. By Frobenius' Theorem a foliation of dimension $n-q$ corresponds to an involutive C^∞-distribution $T\mathcal{F}$ of rank $n-q$ on X.

We call $(X,\mathcal{F})$ a foliated manifold and if $p \in X$ then *the leaf* $\mathcal{L}$ *of* $\mathcal{F}$ *through* p is the subset

$$\mathcal{L} := \{q \in X : \exists \gamma : [0,1] \to X \text{ piecewise smooth from } p \text{ to } q,\ \dot{\gamma}(t) \in T_{\gamma(t)}\mathcal{F}\}.$$

In fact, $\mathcal{L}$ is a $n-q$-dimensional maximal integral (immersed) submanifold of X. An *$\mathcal{F}$-flat coordinate neighborhood* is a chart

$$\varphi : U \to U_x \times U_y \qquad p \mapsto (x^1(p), \ldots, x^{n-q}(p), y^1(p), \ldots, y^q(p))$$

such that the leaves are locally given by $\{y^1 = \text{const}, \ldots, y^q = \text{const}\}$ and U_x, U_y are products of open intervals. We say $\mathcal{F}$ is *regular* at $p \in X$ if there is an $\mathcal{F}$-flat coordinate neighborhood around p whose intersection with each leaf of $\mathcal{F}$ is connected or empty.

Next, we review the notion of leaf holonomy. For a more detailed exposition we refer to [BG08, Ch. 2.3.]. Given a leaf $\mathcal{L}$ of $\mathcal{F}$ and $p_1, p_2 \in \mathcal{L}$ let $\gamma : [0,1] \to \mathcal{L}$ be a continuous curve such that $\gamma(0) = p_1$ and $\gamma(1) = p_2$. We can subdivide $[0,1]$ into subintervals $[t_{k-1}, t_k]$ with $0 = t_0 < \ldots < t_\ell = 1$ such that $\gamma([t_{k-1}, t_k])$ is contained in a chart (U_k, φ_k) of a maximal foliated atlas. Moreover, we may pick $\varphi_k : U_k \to \mathbb{R}^{n-q} \times \mathbb{R}^q$ such that the connected component of $\mathcal{L} \cap U_k$ containing $\gamma([t_{k-1}, t_k])$ corresponds under φ_k to $\mathbb{R}^{n-q} \times \{0\}$. If $\tau \in [t_{k-1}, t_k]$ then we call $\varphi_k^{-1}(\{\gamma(\tau)\} \times \mathbb{R}^q)$ a transversal through $\gamma(\tau)$. Fix transversals N_{t_k} through each $\gamma(t_k)$. For each k we derive a diffeomorphism from an open neighborhood $V_{t_{k-1}} \subset N_{t_{k-1}}$ onto an open neighborhood $V_{t_k} \subset N_{t_{k-1}}$ as indicated in the following diagram

$$\begin{tikzcd} V_{t_{k-1}} \arrow[rr] \arrow[dr, "pr_{\mathbb{R}^q} \circ \varphi_{k-1}|_{V_{t_{k-1}}}"'] & & V_{t_k} \arrow[dl, "pr_{\mathbb{R}^q} \circ \varphi_k|_{V_{t_k}}"] \\ & \mathbb{R}^q & \end{tikzcd}$$

Hence, we derive a germ h_γ of a local diffeomorphism from (N_0, p_1) to (N_1, p_2). Suppose now that $p := p_1 = p_2$ and $N_p := N_0 = N_1$. Then h_γ is a germ of a local diffeomorphism of (N_p, p) which does not depend on the intermediate transversals used for its definition. Moreover, h_γ depends only on the class $[\gamma] \in \pi_1(\mathcal{L})$. The set $Hol(\mathcal{L}, N_p)$ of all h_γ is a subgroup of the group of germs of diffeomorphisms of (N_p, p). Since $h_{\gamma * \tilde{\gamma}} = h_{\tilde{\gamma}} \circ h_\gamma$ we derive an epimorphism $\pi_1(\mathcal{L}, p) \twoheadrightarrow Hol(\mathcal{L}, N_p)$. If $\tilde{N}_p$ is another transversal through p we can identify the germs of diffeomorphisms of $(\tilde{N}_p, p)$ with the germs of diffeomorphisms of (N_p, p), i.e., we have an identification of $Hol(\mathcal{L}, N_p)$ with $Hol(\mathcal{L}, \tilde{N}_p)$. If $p \neq \tilde{p} \in \mathcal{L}$

and $N_{\tilde{p}}$ is a transversal through $\tilde{p}$ then $Hol(\mathcal{L}, N_{\tilde{p}}) = \tilde{h}_{\tilde{\gamma}} \circ Hol(\mathcal{L}, N_p) \circ \tilde{h}_{\tilde{\gamma}^{-1}}$ where $\tilde{\gamma}$ is a curve from p to $\tilde{p}$ and $\tilde{h}_{\tilde{\gamma}}$ is a germ of a diffeomorphism from (N_p, p) to $(\tilde{N}_p, \tilde{p})$ constructed using the same technique as above. Therefore, we write $Hol(\mathcal{L})$ instead of $Hol(\mathcal{L}, N_p)$ and call it *the holonomy group of* $\mathcal{L}$.

Definition 1.27. *Let $(X, \mathcal{F})$ be a foliated manifold and write $L_V(\cdot)$ for the Lie derivative along V. A Riemannian metric g on X is bundle-like w.r.t. $\mathcal{F}$ if $(L_V g)(Y_1, Y_2) = 0$ for any open $U \subset X$ and all $V \in \Gamma(U, T\mathcal{F})$, $Y_1, Y_2 \in \Gamma(U, T\mathcal{F}^{\perp_g})$.*

We say $\mathcal{F}$ is a *Riemannian foliation* on X if there exists a Riemannian metric g on X which is bundle-like w.r.t. $\mathcal{F}$ and a *Riemannian flow* is a 1-dimensional oriented Riemannian foliation. If g is bundle-like for $(X, \mathcal{F})$ and $\gamma : I \to X$ is a g-geodesic such that $\dot{\gamma}(0) \in T_{\gamma(0)}\mathcal{F}^\perp$ then $\dot{\gamma}(t) \in T_{\gamma(t)}\mathcal{F}^\perp$ for all $t \in I$. In this case, we say γ is a horizontal geodesic. For the proof of this fact and a comprehensive introduction to Riemannian foliations we refer to [Mol88].

Let $(X, \mathcal{F})$ be a foliated manifold and let ∇ be a connection on $TX/_{T\mathcal{F}}$. We say ∇ is a *basic connection* if $\nabla_V \bar{Y} = pr_{TX/_{T\mathcal{F}}}([V, Y])$ for any $V \in \Gamma(X, T\mathcal{F})$, $\bar{Y} \in \Gamma(X, TX/_{T\mathcal{F}})$ and $Y \in \Gamma(X, TX)$ such that $\bar{Y} = pr_{TX/_{T\mathcal{F}}} Y$. Moreover, we say $(X, \mathcal{F})$ admits a transverse G-structure if there exists a basic connection ∇ such that $Hol(\nabla) \subset G$ (cf. [Mol88] and [Con74]).

Definition 1.28. *Let $(X, \mathcal{F})$ be a Riemannian foliation with a bundle-like metric g on $(X, \mathcal{F})$. The transverse Levi-Civita connection ∇^T on the normal bundle $T\mathcal{F}^\perp$ is given by*

$$\nabla^T_V Y := \begin{cases} \pi_{T\mathcal{F}^\perp}(\nabla^g_V Y) & V \in T\mathcal{F}^\perp, \\ \pi_{T\mathcal{F}^\perp}([V, Y]) & V \in T\mathcal{F}, \end{cases}$$

where $Y \in \Gamma(U, T\mathcal{F}^\perp)$ and $\pi_{T\mathcal{F}^\perp}$ is the orthogonal projection onto $T\mathcal{F}^\perp$.

Given a Riemannian foliation $(X, \mathcal{F})$ we can consider the exact sequence $0 \to T\mathcal{F} \to TX \to TX/_{T\mathcal{F}}$. A bundle-like metric g induces a Riemannian metric $g^T(\cdot, \cdot) := g(\pi^g_{T\mathcal{F}^\perp}(\cdot), \pi^g_{T\mathcal{F}^\perp}(\cdot))$ on $TX/_{T\mathcal{F}}$. Moreover, ∇^T induces a basic connection on $TX/_{T\mathcal{F}}$ and if $\tilde{g}$ is another bundle-like metric on $(X, \mathcal{F})$ then its induced connections on $TX/_{T\mathcal{F}}$ coincide if $g^T = \tilde{g}^T$ (cf. [Mol88, Lemma 3.3]). For a foliated manifold $(X, \mathcal{F})$ let $X/_{\mathcal{F}}$ be its set of leaves equipped with the quotient topology w.r.t. the map

$$\pi : X \to X/_{\mathcal{F}}, \qquad p \mapsto (\text{leaf through } p).$$

In general, the space of leaves $X/_{\mathcal{F}}$ is not Hausdorff. However, we have

Theorem 1.29 ([Rei61], [Her60, Thm. 4.4], [Esc82, Thm. 2.2])**.** *Let $(X, \mathcal{F})$ be a foliated manifold and π as above. If all leaves of $\mathcal{F}$ are closed subsets of X and if there exists a complete bundle-like metric on $(X, \mathcal{F})$ then*

1. *the holonomy of each leaf is finite and $X/_{\mathcal{F}}$ admits the structure of a Riemannian orbifold such that π is a Riemannian orbifold submersion.*

2. *$X/_{\mathcal{F}}$ admits the structure of a smooth manifold and a complete Riemannian metric $g^{\mathcal{F}}$ making π a Riemannian submersion if all leaves have trivial leaf holonomy.* □

Theorem 1.30 ([Mol88, Prop. 3.7])**.** *Let $(X, \mathcal{F})$ be a foliated manifold such that all leaves of $\mathcal{F}$ are compact. If $(X, \mathcal{F})$ admits a bundle-like Riemannian metric then $X/_{\mathcal{F}}$ admits the structure of a Riemannian orbifold such that π is Riemannian orbifold submersion. In particular, if all leaves have trivial leaf holonomy then π is a smooth Riemannian fiber bundle.* □

Orbifolds are used only once in this presentation. Therefore, we refer to [BG08, Ch. 4] for its definition and applications to foliation theory. We will apply

Theorem 1.31 (Carrière [BG08, Thm. 2.6.4])**.** *Let $(X, \mathcal{F})$ be a Riemannian flow on the compact manifold X. If $\mathcal{L}$ is a leaf of $\mathcal{F}$ then its closure $\bar{\mathcal{L}}$ in X is diffeomorphic to a torus.* □

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. On a foliated manifold $(X, \mathcal{F})$ we define the sheaves $\mathcal{A}^k_{\mathcal{F},\mathbb{K}}$ by

$$\mathcal{A}^k_{\mathcal{F},\mathbb{K}}(U) := \{\omega \in \mathcal{A}^k_{X,\mathbb{K}}(U) : V \lrcorner \omega = L_V \omega = 0,\ \forall V \in \Gamma(U, T\mathcal{F})\}.$$

We refer to any $\omega \in \Gamma(X, \mathcal{A}^k_{\mathcal{F},\mathbb{K}})$ as a *basic k-form on* $(X, \mathcal{F})$. Note, that $\bigoplus_k \mathcal{A}^k_{\mathcal{F},\mathbb{K}}(X)$ is closed under addition and exterior multiplication. Moreover, $L_V d\omega = dL_V\omega = 0$ and $V \lrcorner d\omega = L_V\omega - d(V \lrcorner \omega) = 0$ for a basic form, i.e., we derive a subcomplex of the de Rham complex.

Definition 1.32. *The basic cohomology ring $H^*_B(X, \mathcal{F}, \mathbb{K})$ is given by cohomology of the complex $(\mathcal{A}^k_{\mathcal{F},\mathbb{K}}(X), d|_{\mathcal{A}^k_{\mathcal{F},\mathbb{K}}(X)})$ with the wedge product inducing the ring structure. Moreover, we write $H^*_B(X, \mathcal{F})$ instead of $H^*_B(X, \mathcal{F}, \mathbb{R})$.*

Suppose g is bundle-like for the Riemannian flow $(X, \mathcal{F})$ and let $V \in \Gamma(X, TX)$ be a g-unit length vector field generating $\mathcal{F}$. We define the *mean curvature* 1*-form* by $\kappa_g := g(pr_{T\mathcal{F}^\perp}(\nabla^g_V V), \cdot)$.

Definition 1.33. *Let $(X, \mathcal{F})$ be a Riemannian flow and let $V \in \Gamma(X, TX)$ be a nowhere vanishing vector field generating $\mathcal{F}$. We say $(X, \mathcal{F})$ is an isometric Riemannian flow if there exists a bundle-like Riemannian metric g on $(X, \mathcal{F})$ such that V is a g-Killing vector field of unit length, i.e., $\kappa_g = 0$.*

More generally, a Riemannian foliation is said to be *taut* if there is a bundle-like metric such that all of its leaves are minimal submanifolds. Hence, a Riemannian flow is isometric if it is taut.

Given a bundle-like metric g on the Riemannian flow $(X, \mathcal{F})$ the *basic Laplace operator* is given by $\Delta_B = \Delta^g_B := d\delta_B + \delta_B d : \mathcal{A}^k_{\mathcal{F}}(X) \to \mathcal{A}^k_{\mathcal{F}}(X)$, where δ_B is the L^2-adjoint of $d|_{\mathcal{A}^k_{\mathcal{F}}(X)}$. For a more detailed introduction to basic global analysis we refer to [HR10], [PR96] and the references therein.

Theorem 1.34 (Domínguez [Dom98] and Mason [Mas00])**.** *If $(X, \mathcal{F})$ is a Riemannian flow on the compact manifold X and g is a bundle-like w.r.t. $(X, \mathcal{F})$ then there exists a bundle-like metric $\tilde{g}$ on X such that $\kappa_{\tilde{g}} \in \mathcal{A}^1_{\mathcal{F}}(X)$, $\Delta_B^{\tilde{g}} \kappa_{\tilde{g}} = 0$ and $\tilde{g}^T = g^T$.* □

We say the foliated manifold $(X, \mathcal{F})$ with $q := \operatorname{codim} \mathcal{F}$ is *transversely orientable* if it admits a transverse $GL^+(q)$-structure.

Definition 1.35. *Let $(X, \mathcal{F})$ be a Riemannian flow on the compact manifold X and let g be bundle-like w.r.t. $(X, \mathcal{F})$ such that $\kappa_g \in \mathcal{A}^1_{\mathcal{F}}(X)$ is basic-harmonic.*

1. *The Álvarez-class is given by $[\kappa_g] \in H^1_B(X, \mathcal{F})$.*
2. *The dual basic cohomology $H^*_{d-\kappa_g}(X, \mathcal{F})$ is the cohomology of the complex $(\mathcal{A}^k_{\mathcal{F}}(X), d|_{\mathcal{A}^k_{\mathcal{F}}(X)} - \kappa_g \wedge \cdot)$.*

Theorem 1.36 (Transverse Hodge Theorem [PR96])**.** *Let $(X, \mathcal{F})$ be a transversely oriented Riemannian flow on the compact oriented manifold X and let g be bundle-like w.r.t. $(X, \mathcal{F})$. Define $\mathcal{H}^k(X, \mathcal{F}, g) := \{\alpha \in \mathcal{A}^k_{\mathcal{F}}(X) : \Delta^g_B \alpha = 0\}$. Then*

1. *$\mathcal{H}^k(X, \mathcal{F}, g)$ is finite dimensional.*
2. *For any class $\varphi \in H^k_B(X, \mathcal{F})$ there is a unique representative $\alpha \in \varphi$ such that $\alpha \in \mathcal{H}^k(X, \mathcal{F}, g)$.* □

Proposition 1.37 ([HR10])**.** *Let $(X, \mathcal{F})$ be a transversely oriented Riemannian flow on the compact manifold X and let g be bundle-like w.r.t. $(X, \mathcal{F})$ such that $\kappa_g \in \mathcal{A}^1_{\mathcal{F}}(X)$ is basic-harmonic. Then $H^{\dim X - 1}_B(X, \mathcal{F}) \in \{0, \mathbb{R}\}$ and $H^i_B(X, \mathcal{F}) \cong H^{\dim X - 1 - i}_{d-\kappa_g}(X, \mathcal{F})$ for all $i \geq 0$. Moreover, the following are equivalent:*

1. *The basic cohomology groups satisfy Poincaré duality.*
2. *$H^{\dim X - 1}_B(X, \mathcal{F}) = \mathbb{R}$.*
3. *The Álvarez-class vanishes.*
4. *$(X, \mathcal{F})$ is taut.* □

Finally, we summarize some needed results about codimension one foliations.

Proposition 1.38 ([Con74])**.** *Let $(X, \mathcal{F})$ be a foliated manifold and $Z \in \Gamma(X, TX)$ a complete vector field such that $TX/_{T\mathcal{F}} = span\{pr_{TX/_{T\mathcal{F}}}(Z)\}$ and $[T\mathcal{F}, Z] \subset T\mathcal{F}$. Let $\mathcal{L}$ be a leaf of $\mathcal{F}$.*

1. *If there is no leaf of $\mathcal{F}$ which is closed in X then each leaf is dense in X.*
2. *If there is a leaf which is closed in X then $X \to X/_{\mathcal{F}}$ is a smooth fiber bundle and $X/_{\mathcal{F}} \in \{\mathbb{R}, S^1\}$.*
3. *We have $\tilde{X} = \tilde{\mathcal{L}} \times \mathbb{R}$, where $\tilde{X}$, $\tilde{\mathcal{L}}$ denote the universal covers of X, $\mathcal{L}$.*
4. *The inclusion $\mathcal{L} \to X$ induces a monomorphism $\pi_1(\mathcal{L}) \to \pi_1(X)$ onto a normal subgroup. If X is compact then $\pi_1(X)/_{\pi_1(\mathcal{L})} = \mathbb{Z}^r$ for some $r \geq 1$ and $r = 1$ if and only if $\mathcal{L}$ is closed in X.*

□

2 Geometry and Topology of Special Lorentzian Manifolds

2.1 Weakly Irreducible Groups in O(r,s) and Screen Bundles

The purpose of this section is to introduce the screen bundle and the screen tree. Moreover, we study the relation of the screen holonomy to the (full) holonomy of the underlying pseudo-Riemannian vector bundle.

Let $1 \leq r \leq p$ as well as $q \geq 2(p-r)$ and consider a weakly irreducible subalgebra $\mathfrak{h} \subset \mathfrak{so}(p, 2r-p+q)$ with index r.[1] Suppose $q \geq 1$ and let $W \cap W^{\perp}$ be an $\mathfrak{h}$-invariant isotropic subspace of dimension r. We can find a basis $(v_1, \dots, v_r, e_1, \dots, e_q, w_1, \dots, w_r)$ such that $\mathrm{span}(v_1, \dots, v_r) = W \cap W^{\perp}$ and

$$\langle v_i, v_j \rangle = \langle w_i, w_j \rangle = \langle v_i, e_j \rangle = \langle w_i, e_j \rangle = 0, \qquad \langle v_i, w_j \rangle = \delta_{ij}$$

as well as $\langle e_i, e_j \rangle = \varepsilon_i \delta_{ij}$ where

$$\varepsilon_i = \begin{cases} -1 & i \leq p-r, \\ +1 & \text{otherwise.} \end{cases}$$

Therefore, $\mathfrak{h} \subset \mathrm{stab}_{\mathfrak{so}(p,2r-p+q)}(W \cap W^{\perp}) \subset \mathfrak{so}(p, 2r-p+q)$ and with respect to this basis we derive the identification

$$\mathrm{stab}(W \cap W^{\perp}) = \left\{ \begin{pmatrix} & -Y_1^T & \\ A & \vdots & C \\ & -Y_r^T & \\ 0 & B & Y_1 \cdots Y_r \\ 0 & 0 & -A^T \end{pmatrix} : \begin{array}{c} A \in \mathfrak{gl}(r), \\ B \in \mathfrak{so}(p-r, q-p+r), \\ C \in \mathfrak{so}(r), \\ Y_j \in \mathbb{R}^q \end{array} \right\}.$$

We define

$$\mathfrak{g} := pr_{\mathfrak{so}(p-r,q-p+r)}(\mathfrak{h}) \subset \mathfrak{so}(p-r, q-p+r).$$

[1] We focus w.l.o.g. on subalgebras in $\mathfrak{so}(r,s)$ where $r \leq s$. Hence, we need the inequality $q \geq 2(p-r)$ in order to ensure $2r-p+q \geq p$.

Moreover, for $i,j \leq r$ and $R \in \mathcal{K}(\mathfrak{h})$ let $\mathcal{E} := \mathrm{span}\{e_1,\dots,e_q\}$ and

$$\begin{aligned}\mathcal{P}_0^R &:= pr_{\mathcal{E}} \circ R|_{\mathcal{E}\times\mathcal{E}\times\mathcal{E}} \in \Lambda^2\mathcal{E}^* \otimes \mathfrak{g},\\ \mathcal{P}_i^R &:= pr_{\mathcal{E}} \circ R(w_i,\cdot)|_{\mathcal{E}\times\mathcal{E}} \in \mathcal{E}^* \otimes \mathfrak{g},\\ \mathcal{Q}_{ij}^R &:= pr_{\mathcal{E}} \circ R(w_i,w_j)|_{\mathcal{E}} \in \mathfrak{g}.\end{aligned}$$

The Bianchi identity for R implies $\mathcal{P}_0^R \in \mathcal{K}(\mathfrak{g})$ and $\mathcal{P}_i^R \in \mathcal{B}_{\langle\cdot,\cdot\rangle|_{\mathcal{E}}}(\mathfrak{g})$.[2] We will apply the following observation

Lemma 2.1. *Let $1 \leq r \leq p$ and $q \geq 2(p-r)$. If $\mathfrak{h} \subset \mathfrak{so}(p, 2r-p+q)$ is a weakly irreducible Berger algebra with index r then*

$$\mathfrak{g} = span\{\mathcal{P}_0^R(Y_1,Y_2), \mathcal{P}_k^R(Y_k), \mathcal{Q}_{ij}^R : Y_\cdot \in \mathcal{E},\ R \in \mathcal{K}(\mathfrak{h})\}.$$

Proof. Since $\mathfrak{h}$ is a Berger algebra we have $\mathfrak{h} = \mathrm{span}\{R(x,y) : R \in \mathcal{K}(\mathfrak{h}),\ x,y \in \mathbb{R}^{2r+q}\}$. Thus,

$$\mathfrak{g} = \mathrm{span}\{pr_{\mathcal{E}} \circ R(x,y)|_{\mathcal{E}} : R \in \mathcal{K}(\mathfrak{h}),\ x,y \in \mathbb{R}^{2r+q}\}.$$

Fix $\xi_1, \xi_2 \in \mathbb{R}^{2r+q}$ such that $\xi_\ell = \alpha_\ell^j v_j + Y_\ell + \beta_\ell^j w_j$ with $Y_\ell \in \mathcal{E}$. For any $R \in \mathcal{K}(\mathfrak{h})$ and $E \in \mathcal{E}$ we have

$$pr_{\mathcal{E}} \circ R(\xi_1,\xi_2)E = \sum_{k=1}^{q} \varepsilon_k \langle R(\xi_1,\xi_2)E, e_k\rangle e_k.$$

We compute

$$\begin{aligned}\langle R(\xi_1,\xi_2)E,e_k\rangle &= \alpha_1^i\alpha_2^j\langle R(v_i,v_j)E,e_k\rangle + \alpha_1^i\langle R(v_i,Y_2)E,e_k\rangle\\ &\quad + \alpha_2^j\langle R(Y_1,v_j)E,e_k\rangle + \alpha_1^i\beta_2^j\langle R(v_i,w_j)E,e_k\rangle\\ &\quad + \beta_1^i\alpha_2^j\langle R(w_i,v_j)E,e_k\rangle + \beta_1^i\langle R(w_i,Y_2)E,e_k\rangle\\ &\quad + \beta_2^j\langle R(Y_1,w_j)E,e_k\rangle + \beta_1^i\beta_2^j\langle R(w_i,w_j)E,e_k\rangle\\ &\quad + \langle R(Y_1,Y_2)E,e_k\rangle.\end{aligned}$$

Since R is an algebraic curvature tensor with values in $\mathfrak{h} \subset so(p, 2r-p+q)$ we have

$$\langle R(v_i,v_j)E,e_k\rangle = \langle \underbrace{R(E,e_k)v_i}_{\in W\cap W^\perp}, \underbrace{v_j}_{\in W\cap W^\perp}\rangle = 0.$$

The same way we derive $\langle R(v_i,Y_2)E,e_k\rangle = \langle R(Y_1,v_j)E,e_k\rangle = 0$ since $\mathcal{E} \perp W \cap W^\perp$.

[2]For $Y_1,Y_2,Y_3 \in \mathcal{E}$ we have

$$\begin{aligned}&\langle \mathcal{P}_i^R(Y_1)Y_2,Y_3\rangle + \langle \mathcal{P}_i^R(Y_2)Y_3,Y_1\rangle + \langle \mathcal{P}_i^R(Y_3)Y_1,Y_2\rangle\\ &= \langle R(w_i,Y_1)Y_2,Y_3\rangle + \langle R(w_i,Y_2)Y_3,Y_1\rangle + \langle R(w_i,Y_3)Y_1,Y_2\rangle\\ &= -\langle R(Y_2,Y_3)Y_1,w_i\rangle - \langle R(Y_3,Y_1)Y_2,w_i\rangle - \langle R(Y_1,Y_2)Y_3,w_i\rangle\\ &= 0.\end{aligned}$$

Moreover, the Bianchi identity for R implies

$$\begin{aligned}\langle R(v_i, w_j)E, e_k\rangle &= -\langle \underbrace{R(w_j, E)v_i}_{\in W\cap W^\perp}, e_k\rangle - \langle R(E, v_i)w_j, e_k\rangle \\ &= -\langle \underbrace{R(w_j, e_k)E}_{\in W\cap W^\perp \oplus \mathcal{E}}, v_i\rangle = 0\end{aligned}$$

and $\langle R(w_i, v_j)E, e_k\rangle = 0$. Thus,

$$pr_{\mathcal{E}} \circ R(\xi_1, \xi_2)E = \mathcal{P}_0(Y_1, Y_2)E + \mathcal{P}_j(\beta_1^j Y_2 - \beta_2^j Y_1)E + \beta_1^i \beta_2^j \mathcal{Q}_{ij} E.$$

□

The first part of the next result is due to Leistner in case that $p = 1$ while the second part is a Corollary to [Gal04].

Corollary 2.2. *Let $1 \le r \le p$ and $q \ge 2(p-r)$, $q \ge 1$. If $\mathfrak{h} \subset \mathfrak{so}(p, 2r - p + q)$ is a weakly irreducible Berger algebra with index r then*

- $\mathfrak{g}$ *is a weak Berger algebra having the Borel-Lichnérowicz property if $r = 1$*
- *there are counterexamples to this statement if $r > 1$.*

Proof. If $r = 1$ we have $\mathcal{Q}_{ij} = 0$ and Lemma 2.1 implies

$$\mathfrak{g} = \mathrm{span}\{\mathcal{P}_0^R(Y_1, Y_2), \mathcal{P}_k^R(Y_k) : Y_\cdot \in \mathcal{E},\ R \in \mathcal{K}(\mathfrak{h})\}.$$

However, the footnote following Definition 1.4 implies $\mathcal{P}_0^R(Y_1, \cdot) \in \mathcal{B}_{\langle\cdot,\cdot\rangle|_{\mathcal{E}}}(\mathfrak{g})$. Since $\mathcal{P}_k^R \in \mathcal{B}_{\langle\cdot,\cdot\rangle|_{\mathcal{E}}}(\mathfrak{g})$ we conclude the statement.

In [Gal04] Galaev constructed pseudo-Riemannian metrics on $\mathbb{R}^{n+4}$ of signature $(2, n+2)$ whose holonomy algebra is weakly irreducible with index 2. For these algebras any (irreducible) subalgebra of $\mathfrak{so}(n)$ can appear as $\mathfrak{g}$. However, by Leistner's theorem 1.9 this is not the case for weak Berger algebras in $\mathfrak{so}(n)$. □

Remark 2.3. As we have seen in Corollary 2.2 even if $\mathfrak{h}$ is the holonomy algebra of a pseudo-Riemannian manifold $\mathfrak{g}$ is not necessarily a weak Berger algebra if $r > 1$. However, we will define a certain class of submanifolds in pseudo-Riemannian spaces of constant curvature for which we prove in Proposition 3.12 that the algebra $\mathfrak{g}$ associated to the normal holonomy representation is still a (weak) Berger algebra if $r > 1$. □

Let (E, h, ∇^E, π, X) be a pseudo-Riemannian vector bundle whose full holonomy group $Hol(\nabla^E) \subset O(p, 2r - p + q)$ is weakly irreducible with index r where we assume $1 \le r \le p$ and $1 \le q \ge 2(p-r)$. Let $W \cap W^\perp$ be a $Hol(\nabla^E)$-invariant isotropic subspace of dimension r. By the holonomy principle we have an isotropic subbundle $\Xi \subset E$ corresponding to $W \cap W^\perp$. Additionally, the orthogonal complement $\Xi^\perp \subset E$ is a ∇^E-parallel subbundle.

Consider the vector bundle $\mathcal{S} := \mathrm{Coker}(\Xi \hookrightarrow \Xi^\perp)$. Clearly, we have an induced metric $h^{\mathcal{S}}$ and an induced connection $\nabla^{\mathcal{S}}$ on $\mathcal{S}$. It is not difficult to see that $h^{\mathcal{S}}$ has signature $(p-r, q-p+r)$ and $\nabla^{\mathcal{S}} h^{\mathcal{S}} = 0$.

Definition 2.4. *For any pseudo-Riemannian vector bundle (E, h, ∇^E, π, X) whose full holonomy group $Hol(\nabla^E) \subset O(p, 2r-p+q)$ is weakly irreducible with index r the pseudo-Riemannian vector bundle $(\mathcal{S}, h^{\mathcal{S}}, \nabla^{\mathcal{S}}, \pi^{\mathcal{S}}, X)$ is called the (canonical) screen bundle of (E, h, ∇^E, π, X).*

More generally, to any pseudo-Riemannian vector bundle (E, h, ∇^E, π, X) over a simply connected base X we associate a directed, rooted tree where each vertex is the representation of a holonomy algebra. More precisely, we define the root to be the holonomy representation $\mathfrak{hol}(\nabla^E)$. If $\mathfrak{hol}(\nabla^E)$ is reducible admitting a Borel-Lichnérowicz decomposition $\mathfrak{hol}(\nabla^E) = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_\ell$ then we define each irreducible representation $\mathfrak{h}_i$ to be a child of the root.

Moreover, if $\mathfrak{h}_j$ is weakly irreducible with positive index and $q \geq 1$ we define the representation of the holonomy algebra of the associated screen bundle $(\mathcal{S}_j, h^{\mathcal{S}_j}, \nabla^{\mathcal{S}_j}, \pi^{\mathcal{S}_j}, X)$ to be a child of the root. In case $q = 0$ there is no screen bundle of non-vanishing rank and we attach a trivial child to the root.

Any irreducible or trivial vertex and any vertex not admitting a Borel-Lichnérowicz decomposition is defined to be a leaf. Inductively, we derive a finite rooted tree $\mathcal{T}(\nabla^E)$ by considering $(\mathcal{S}_j, h^{\mathcal{S}_j}, \nabla^{\mathcal{S}_j}, \pi^{\mathcal{S}_j}, X)$ as a pseudo-Riemannian vector bundle.

Definition 2.5. *Let (E, h, ∇^E, π, X) be a pseudo-Riemannian vector bundle and $F : \tilde{X} \to X$ the universal covering of X. An associated finite rooted tree $\mathcal{T}(\nabla^{F^*E})$ is called a screen tree and we say (E, h, ∇^E, π, X) admits a complete screen tree if all leaves are represented by irreducible or trivial representations.*

In order to study the geometry of $(\mathcal{S}, \nabla^S)$ it is convenient to study a non-canonical realization of $\mathcal{S}$ as a subbundle in E which is isomorphic (as a pseudo-Riemannian vector bundle) to $\mathcal{S}$ and given by a non-canonical splitting $s : \mathcal{S} \to \Xi^\perp$ of the exact sequence[3]

$$0 \to \Xi \to \Xi^\perp \to \mathcal{S} \to 0.$$

We define $S := s(\mathcal{S})$ and call it a (non-canonical) realization of $\mathcal{S}$. The connection ∇^E on E induces connections on the subbundles Ξ and S given by

$$\nabla^\Xi := pr_\Xi \circ \nabla^E|_\Xi \qquad \text{and} \qquad \nabla^S := pr_S \circ \nabla^E|_S.$$

The canonical bundle morphism $S \xrightarrow{F} \mathcal{S}$ is a vector bundle isomorphism such that $\nabla^S = F^*\nabla^{\mathcal{S}}$ and $h|_{S\times S} = F^* h^{\mathcal{S}}$, i.e., $Hol(S, \nabla^S) = Hol(\mathcal{S}, \nabla^{\mathcal{S}})$. Moreover, $S^\perp \subset E$ has signature (r, r) and $\Xi \subset S^\perp$. If $r = 1$ and $p \in X$ the light cone in $S_p^\perp$ is the union of two lines one of which is given by Ξ_p. Hence, we derive

[3] As we assumed all manifolds to be second countable Hausdorff spaces the vector bundles $\Xi, \Xi^\perp, \mathcal{S}$ admit partitions of unity ensuring the existence of a splitting s.

Corollary 2.6. *Let (E, h, ∇^E, π, X) be a pseudo-Riemannian vector bundle whose full holonomy group $Hol(\nabla^E) \subset O(p, 2r-p+q)$ acts weakly irreducibly with index $r = 1$ and let $S \subset E$ be a realization of the screen bundle.*

Then there is a uniquely defined isotropic subbundle $\Theta \subset S^\perp$ of rank one with the following property: If $V \in \Gamma(U \subset X, \Xi)$ then there exists a unique section $Z \in \Gamma(U \subset X, \Theta)$ such that $h(V, Z) = 1$. □

A similar statement as in Corollary 2.6 clearly fails if $r > 1$ as one may check using simple linear algebra.

Next, we study the relation of $Hol(\mathcal{S}, \nabla^{\mathcal{S}})$ and the orthogonal part

$$G = pr_{O(p-r,q-p+r)}(Hol(\nabla^E)).$$

This has been done for the corresponding Lie algebras in case $Hol(\nabla^E) \subset O(1, q+1)$ by Leistner in [Lei06]. In order to study submanifolds and the topology of Lorentzian manifolds we need the result for higher signatures and the full holonomy group.

Proposition 2.7. *For any pseudo-Riemannian vector bundle (E, h, ∇^E, π, X) whose full holonomy group $Hol(\nabla^E) \subset O(p, 2r-p+q)$ acts weakly irreducibly with index r the holonomy representation of any non-canonical realization (S, ∇^S) of the screen bundle is isomorphic to the induced representation $G = pr_{O(p-r,q-p+r)}(Hol(\nabla^E))$.*

Proof. Consider a smooth curve $\gamma : [0,1] \to X$ and $W_0 \in \Xi^\perp_{\gamma(0)} \subset E_{\gamma(0)}$. Let W_t denote the ∇^E-parallel displacement of W_0 along γ. First, we assume there exists an open neighborhood $\gamma([0,1]) \subset U \subset X$ and two sets of r linearly independent sections $(V_1, \ldots, V_r)$ and $(Z_1, \ldots, Z_r)$ on U such that $\Xi = \operatorname{span}\{V_1, \ldots, V_r\}$, $h(Z_i, Z_j) = h(Z_i, V_j) = 0$ for $i \neq j$, $h(Z_i, V_i) = 1$ and $Z_i \in S^\perp$. Additionally, we assume there are q linearly independent sections $(e_1, \ldots, e_q)$ such that $S = \operatorname{span}\{e_1, \ldots, e_q\}$ and $h(e_i, e_j) = \varepsilon_i \delta_{ij}$. We may write $W_t = W_t^i V_i(t) + W_t^{i+r} e_i(t) + W_t^{i+q+r} Z^i(t)$. Thus,

$$\dot{W}_t^i + \sum_j \omega_j^i(\dot{\gamma}(t)) W_t^j = 0.$$

Since $W_0 \in \Xi^\perp$ and $\Xi^\perp$ is ∇^E-parallel the holonomy principle (Thm. 1.11) implies $W_t \in \Xi^\perp$, i.e., $W^{i+q+r} = 0$ for $1 \leq i \leq r$. In particular, we have $\dot{W}_t^{i+r} = -\sum_{j=1}^{q+r} \omega_j^{i+r}(\dot{\gamma}(t)) W_t^j$ for $1 \leq i \leq q$. Moreover, $\omega_j^{i+r}(\dot{\gamma}(t)) = h(\nabla^E_{\dot{\gamma}(t)} V_j, e_i) = 0$ for $j \leq r$. For $1 \leq j \leq q$ we have

$$\omega_{j+r}^{i+r}(\dot{\gamma}(t)) = h(\nabla^E_{\dot{\gamma}(t)} e_j, e_i) = h(\nabla^S_{\dot{\gamma}(t)} e_j, e_i) = \omega_j^{i\,\nabla^S}(\dot{\gamma}(t)).$$

Hence, $h(W_t, e_i) = h(\tilde{W}_t, e_i)$ where $\tilde{W}_t$ is the ∇^S-parallel displacement of $pr_S(W_0)$ along γ. For the general case we can find a finite covering of $[0,1]$ by closed subintervals $[a_k, b_k]$ such that $\gamma|_{[a_k,b_k]}$ has the above properties. □

Remark 2.8. For a general Lorentzian manifold (X, g) admitting a lightlike nowhere vanishing vector field $V \in \Gamma(X, TX)$ such that $\nabla.V \in \mathbb{R} \cdot V$ we define $\Xi := \operatorname{span}\{V\}$ and its orthogonal complement $\Xi^\perp$. The quotient bundle $\mathcal{S} := \operatorname{Coker}(\Xi \hookrightarrow \Xi^\perp)$, its non-canonical realizations in X and its induced connection $\nabla^{\mathcal{S}}$ are defined in the same way as the screen bundle. Moreover, Cor. 2.6 and Prop. 2.7 naturally generalize to this situation. Hence, we will still refer to $\mathcal{S}$ and $Hol(\nabla^{\mathcal{S}})$ as the screen bundle resp. holonomy of (X,g) $\square$

Using Corollary 2.2 and Proposition 2.7 we derive

Corollary 2.9. *Let (E,h,∇^E,π,X) be a pseudo-Riemannian vector bundle with good holonomy over a simply connected base X. If $Hol(\nabla) \subset SO_0(p,2-p+q)$ is weakly irreducible with index $r=1$ then $(\mathcal{S},h^{\mathcal{S}},\nabla^{\mathcal{S}},\pi^{\mathcal{S}},X)$ is a vector bundle with almost good holonomy $Hol(\nabla^S) \subset SO_0(p-1,q-p+1)$.* $\square$

The next Lemma ensures the global existence of a codimension one foliation on a Lorentzian manifold with weakly irreducible holonomy algebra and is useful when studying submanifolds in spaces of constant curvature.

Lemma 2.10. *Let (E,h,∇^E,π,X) be a pseudo-Riemannian vector bundle and $p \in X$ such that $\mathfrak{hol}_p(\nabla^E)$ admits a Borel-Lichnérowicz decomposition of the form*

$$E_p = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \ldots \oplus \mathcal{E}_\ell \quad \textit{and} \quad \mathfrak{hol}_p(\nabla^E) = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_\ell$$

where $\dim \mathcal{E}_j > 1$ for all $j \geq 1$, $\mathfrak{h}_2,\ldots,\mathfrak{h}_\ell$ act irreducibly and $\mathcal{E}_0$ is definite. Suppose $\mathfrak{h}_1 \subset \mathfrak{so}(\mathcal{E}_1,h|_{\mathcal{E}_1})$ acts weakly irreducibly with index 1 such that $\mathfrak{h}_1 \cdot v \subset \mathbb{R}\cdot v$ for some $0 \neq v \in \mathcal{E}_1$. If $\mathfrak{h}_1 \neq \mathfrak{so}(1,1)$ then $Hol_p(\nabla^E)\cdot v \subset \mathbb{R}\cdot v$.

Proof. Let H_i be the connected Lie subgroup of $H := Hol^0_p(\nabla^E)$ whose Lie algebra is $\mathfrak{h}_i$. We have to show $g^{-1}(v) \in \mathbb{R}\cdot v$ for all $g \in \mathrm{Norm}_{O(E_p,h_p)}(H)$ since

$$Hol_p(\nabla^E) \subset \mathrm{Norm}_{O(E_p,h_p)}(H) := \{g \in O(E_p,h_p =: \langle\cdot,\cdot\rangle) : g^{-1}Hg = H\}.$$

For any $h \in Hol^0(\nabla^E)$ we have $h(v) = \alpha_h \cdot v$ since H_i acts trivially on $\mathcal{E}_1$ for $i \neq 1$. Therefore, $(g^{-1}hg)(g^{-1}v) = \alpha_h \cdot g^{-1}(v)$ for $g \in \mathrm{Norm}_{O(E_p,h_p)}(Hol^0_p(\nabla^E))$ and $h \in Hol^0_p(\nabla^E)$. For $0 \leq i \leq \ell$ let $\tilde{v}_i \in \mathcal{E}_i$ such that $g^{-1}(v) = \tilde{v}_0 + \ldots + \tilde{v}_\ell$. Using $H_i \subset g^{-1}Hg$ we derive $\mathbb{R}\cdot g^{-1}(v) \ni h \cdot g^{-1}(v) = \tilde{v}_0 + \ldots + \tilde{v}_{i-1} + h\tilde{v}_i + \tilde{v}_{i+1} + \ldots + \tilde{v}_\ell$ for all $h \in H_i$. Hence, $h\tilde{v}_i \in \mathbb{R}\cdot\tilde{v}_i$ and for $i \geq 2$ we conclude $\tilde{v}_i = 0$ since H_i acts irreducibly on $\mathcal{E}_i$. This implies $g^{-1}(v) = \tilde{v}_0 + \tilde{v}_1$.

On the other hand, we have $\mathbb{R}\cdot g^{-1}(v) \ni h\cdot g^{-1}(v) = \tilde{v}_0 + h\tilde{v}_1$ for any $h \in H_1$, i.e., $\tilde{v}_1 \in \mathbb{R}\cdot v$ since H_1 acts weakly irreducibly with index 1 on $\mathcal{E}_1$ and $\mathfrak{h}_1 \neq \mathfrak{so}(1,1)$. If $\mathcal{E}_0 \ni \tilde{v}_0 \neq 0$ we derive the contradiction

$$0 = \langle g^{-1}(v), g^{-1}(v)\rangle = \langle\tilde{v}_0,\tilde{v}_0\rangle + 2\langle\tilde{v}_0,\tilde{v}_1\rangle + \langle\tilde{v}_1,\tilde{v}_1\rangle = \langle\tilde{v}_0,\tilde{v}_0\rangle \neq 0$$

since $\mathcal{E}_0$ is definite. Therefore, $g^{-1}(v) \in \mathbb{R}\cdot v$. $\square$

Let $\mathfrak{h} \subset \mathfrak{so}(1,1+q)$ be a weakly irreducible subalgebra with index 1. Being a subalgebra of a compact Lie algebra its orthogonal projection $\mathfrak{g} = pr_{\mathfrak{so}(q)}(\mathfrak{h}) \subset \mathfrak{so}(q)$ is compact and therefore reductive. Hence, we have $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) + [\mathfrak{g},\mathfrak{g}]$ where $\mathfrak{z}(\mathfrak{g})$ is the center of

$\mathfrak{g}$. Bérard-Bergery and Ikemakhen have shown in [BBI93] that a weakly irreducible subalgebra $\mathfrak{h} \subset \mathrm{stab}_{\mathfrak{so}(1,1+q)}(W \cap W^{\perp}) = (\mathbb{R} \oplus \mathfrak{so}(q)) \ltimes \mathbb{R}^q$ belongs to one of four types.

Theorem 2.11 (Bérard-Bergery & Ikemakhen [BBI93])**.** *A weakly irreducible subalgebra $\mathfrak{h} \subset \mathfrak{so}(1, q+1)$ with index 1 belongs to one of the following types:*

- *Type 1:* $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^q$
- *Type 2:* $\mathfrak{h} = \mathfrak{g} \ltimes \mathbb{R}^q$
- *Type 3:*
$$\mathfrak{h} = \left\{ \begin{pmatrix} \varphi(A) & w^T & 0 \\ 0 & A & -w \\ 0 & 0 & -\varphi(A) \end{pmatrix} : A \in \mathfrak{g},\ w \in \mathbb{R}^q \right\}$$
where $\varphi : \mathfrak{g} \twoheadrightarrow \mathbb{R}$ is an epimorphism satisfying $\varphi|_{[\mathfrak{g},\mathfrak{g}]} = 0$.
- *Type 4: There is $0 < \ell < q$ such that $\mathbb{R}^q = \mathbb{R}^\ell \oplus \mathbb{R}^{q-\ell}$, $\mathfrak{g} \subset \mathfrak{so}(\ell)$ and*
$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & \psi(A)^T & w^T & 0 \\ 0 & 0 & 0 & -\psi(A) \\ 0 & 0 & A & -w \\ 0 & 0 & 0 & 0 \end{pmatrix} : A \in \mathfrak{g},\ w \in \mathbb{R}^\ell \right\}$$
for some epimorphism $\psi : \mathfrak{g} \twoheadrightarrow \mathbb{R}^{q-\ell}$ satisfying $\psi|_{[\mathfrak{g},\mathfrak{g}]} = 0$.

□

Using Leistner's theorem 1.9 we conclude that $\mathfrak{g}$ acts as a Riemannian holonomy representation if $\mathfrak{h} = \mathfrak{hol}_p(X, g)$ is the holonomy algebra of a weakly irreducible, reducible Lorentzian manifold (X, g). Moreover, Galaev has shown in [Gal06] that all possible holonomy groups can be constructed by real analytic Lorentzian metrics on $\mathbb{R}^{q+2}$.

Proposition 2.12. *Let (E, h, ∇^E, π, X) be a pseudo-Riemannian vector bundle of signature $(1, q+1)$ such that $\mathfrak{hol}_p(\nabla^E)$ is weakly irreducible with index 1 for some $p \in X$. Then its full holonomy group $Hol_p(\nabla^E)$ is weakly irreducible with index 1. In particular, if $G \subset SO(q)$ is the connected subgroup corresponding to $\mathfrak{g} := pr_{\mathfrak{so}(q)}(\mathfrak{hol}_p(\nabla^E))$ then*

$$Hol_p(\nabla^E) \subset \left\{ \begin{pmatrix} a & -aY^TB & -\frac{a}{2}Y^TY \\ 0 & B & Y \\ 0 & 0 & \frac{1}{a} \end{pmatrix} : \begin{array}{c} a \in \mathbb{R} \setminus \{0\}, \\ Y \in \mathbb{R}^q, \\ B \in Norm_{O(q)}(G) \subset O(q) \end{array} \right\},$$

where $Norm_{O(q)}(G) := \{g \in O(q) : g^{-1}Gg = G\}$ denotes the normalizer of G in $O(q)$.

Proof. The first part is a special case of Lemma 2.10. For the second statement we consider (E, h, ∇^E, π, X) and its pullback $(F^*E, F^*h, F^*\nabla^E, F^*\pi, \tilde{X})$ where $F : \tilde{X} \to X$ denotes the universal covering of X. If $(\mathcal{S}, h^{\mathcal{S}}, \nabla^{\mathcal{S}}, \pi^{\mathcal{S}}, X)$ is the screen bundle of E then we identify $(F^*\mathcal{S}, F^*h^{\mathcal{S}}, F^*\nabla^{\mathcal{S}}, F^*\pi^{\mathcal{S}}, \tilde{X})$ with the screen bundle of F^*E. Thus, Proposition 2.7 implies $G = Hol(F^*\mathcal{S}, F^*\nabla^{\mathcal{S}}) = Hol^0(\mathcal{S}, \nabla^{\mathcal{S}})$ and we conclude $Hol(\mathcal{S}, \nabla^{\mathcal{S}}) \subset \mathrm{Norm}_{O(q)}(G) \subset O(q)$. □

Remark 2.13. By Leistner's theorem 1.9 $G \subset SO(q)$ acts as a Riemannian holonomy representation if (E, h, ∇^E, π, X) has signature $(1, q+1)$, good holonomy and weakly irreducible holonomy algebra with index 1. We refer to [Bes87, 10.112] and [McI91] for the normalizers of these groups. □

Lemma 2.14. *Let (E, h, ∇^E, π, X) be a pseudo-Riemannian vector bundle of signature $(1, q+1)$ and $\mathbb{R} \cdot v \in E_p$ a $Hol_p(\nabla^E)$-invariant isotropic line. If Ξ is the corresponding parallel isotropic subbundle then*

1. *Ξ admits a global nowhere vanishing section if and only if*

$$pr_{\mathbb{R}\cdot v} \circ Hol(\nabla^E)|_{\mathbb{R}\cdot v} \subset \mathbb{R}_+ \subset \mathbb{R} \setminus \{0\}.$$

2. *Ξ admits a global nowhere vanishing section if and only if (E, h) is time-orientable.*[4]

3. *If $\mathfrak{hol}(\nabla^E)$ is weakly irreducible then it is of type 2 or 4 if and only if there is an open covering $X = \bigcup_k U_k$ and ∇^E-parallel sections $V^{U_k} \in \Gamma(U_k, \Xi)$.*

4. *There exists a $Hol_p(\nabla^E)$-invariant isotropic vector if (E, h) is time-orientable and there is a covering $X = \bigcup_k U_k$ and local parallel sections $V^{U_k} \in \Gamma(U_k, \Xi)$ such that $\check{H}^1(\{U_k\}, \mathbb{R}) = 0$.*

Proof. If $pr_{\mathbb{R}\cdot v} \circ Hol(\nabla^E)|_{\mathbb{R}\cdot v} \subset \mathbb{R}_+$ then $\mathbb{R}_+ \cdot v$ provides a well defined field of directions Ξ_+ in Ξ. Thus, we can find a covering $X = \bigcup_k U_k$ and local sections $V_k \in \Gamma(U_k, \Xi)$ such that $V_k \in \Xi_+$. Using a partition of unity we derive a global section of Ξ. Conversely, by the virtue of the proof of Corollary 2.6 the existence of a global section $V \in \Gamma(X, \Xi)$ implies the existence of a complementary section $Z \in \Gamma(X, \Theta)$. For any curve $\gamma : [0, 1] \to X$ we conclude $h(Z, \tau_\gamma^{\nabla^E}(V_{\gamma(0)})) = h(Z, \alpha(t)V_{\gamma(t)}) = \alpha(t)$. By continuity of α we have $\alpha(1) > 0$, i.e., $pr_{\mathbb{R}\cdot v} \circ Hol(\nabla^E)|_{\mathbb{R}\cdot v} \subset \mathbb{R}_+$.

If (E, h) is time-orientable we may locally choose future pointing sections $V^{U_\alpha} \in \Gamma(U_\alpha, \Xi)$ and use a partition of unity. Conversely, if Ξ admits a global nowhere vanishing section V, so does Θ. If $Z \in \Gamma(X, \Theta)$ denotes this section then $s := \frac{1}{\sqrt{2}}(V - Z)$ is a global section of E such that $h(s, s) = -1$.

For the third statement the existence of the desired covering follows from the holonomy principle. For the converse we may assume X to be simply connected and (E, h) to be time-orientable as we consider $Hol^0(\nabla^E)$. If $V \in \Gamma(X, \Xi)$ is nowhere vanishing then $V|_{U_\alpha} = \lambda^{U_\alpha} V^{U_\alpha}$ and

$$\nabla^E_\cdot V|_{U_\alpha} = d(\log(\lambda^{U_\alpha}))(\cdot)\lambda^{U_\alpha} V^{U_\alpha} = d(\log(\lambda^{U_\alpha}))(\cdot) V|_{U_\alpha}.$$

W.l.o.g. we have $\lambda^{U_\alpha} > 0$ or we replace V^{U_α} by $-V^{U_\alpha}$ if necessary. In particular, the last equation implies $d(\log(\lambda^{U_\alpha})) = d(\log(\lambda^{U_\beta}))$ on $U_\alpha \cap U_\beta$ and therefore $\nabla_\cdot V = \alpha(\cdot)V$ for some closed global 1-form $\alpha = df$ on X. Hence, $e^{-f}V \in \Gamma(X, \Xi)$ is a global nowhere vanishing parallel section.

[4]Here we say (E, h) is time-orientable if there exists a global section $s \in \Gamma(X, E)$ such that $h(s, s) = -1$.

For the last part we need to construct a function $f \in C^\infty(X)$ such that $\alpha = df$. This is standard and can be done by chasing through the following commutative diagram:

$$
\begin{array}{ccccc}
\mathbb{R} & \xrightarrow{\delta_0} & \check{C}^0(\{U_k\},\mathbb{R}) & \xrightarrow{\delta_1} & \check{C}^1(\{U_k\},\mathbb{R}) \\
\downarrow i & & \downarrow i & & \downarrow i \\
\mathcal{A}^0(X) & \xrightarrow{\delta_0} & \check{C}^0(\{U_k\},\mathcal{A}^0) & \xrightarrow{\delta_1} & \check{C}^1(\{U_k\},\mathcal{A}^0) \\
\downarrow d & & \downarrow d & & \\
\mathcal{A}^1(X) & \xrightarrow{\delta_0} & \check{C}^0(\{U_k\},\mathcal{A}^1) & &
\end{array}
$$

More precisely, the local functions $\{U_\alpha, \log(\lambda^U_\alpha)\}_\alpha$ define a Čech cochain in $\check{\mathcal{C}}^0(\{U_k\},\mathcal{A}^0)$. Then $\{U_\alpha \cap U_\beta, \log(\lambda^U_\alpha) - \log(\lambda^U_\beta)\}_{\alpha\beta} = \delta_1(\{U_\alpha, \log(\lambda^U_\alpha)\}_\alpha) \in i(\check{\mathcal{C}}^1(\{U_k\},\mathbb{R}))$ since $d\log(\lambda^U_\alpha) = d\log(\lambda^U_\beta)$ on $U_\alpha \cap U_\beta$. Hence, we derive $\{U_\alpha \cap U_\beta, \log(\lambda^U_\alpha) - \log(\lambda^U_\beta)\}_{\alpha\beta} \in \check{Z}^1(\{U_k\},\mathbb{R})$. If this cocycle is a coboundary there exists constants $(U_\alpha, c_\alpha) \in \check{\mathcal{C}}^0(\{U_k\},\mathbb{R})$ such that $\log(\lambda^U_\alpha) + c_\alpha = \log(\lambda^U_\beta) + c_\beta =: f$. Thus, f is the desired function. $\square$

If (E, h, ∇^E, π, X) is time-orientable and $Hol^0(\nabla^E)$ is weakly irreducible with index 1 leaving an isotropic vector invariant we have a global non-vanishing section $V \in \Gamma(X, \Xi)$ such that $\nabla^E_\cdot V = \alpha(\cdot)V$. Locally $\alpha = d(\log(\lambda^{U_\alpha}))$, i.e., α is closed and induces the same cohomology class as in the Lemma. In particular, the cohomology class of α does not depend on the choice of the global section as two of them differ by a nowhere vanishing function. Hence, $pr_{\mathbb{R}\cdot v} \circ Hol(\nabla^E)|_{\mathbb{R}\cdot v}$ is trivial if and only if $0 = [\alpha] \in H^1(X,\mathbb{R})$. This way we derive a "characteristic" class for any time-orientable Lorentzian manifold for which $\mathfrak{hol}(\nabla^E)$ has a weakly irreducible part of type 2 or 4.[5]

Suppose $Hol^0(X, g)$ is weakly irreducible with index 1, i.e., around any $p \in X$ we have a locally defined recurrent lightlike vector field V. It has been shown in [Wal50] that we can find local coordinates $(x, y^1, \ldots, y^n, z)$ in $U \ni p$ such that

$$g = 2dxdz + 2u_\alpha dy^\alpha dz + f dz^2 + g_{\alpha\beta} dy^\alpha dy^\beta$$

and $\frac{\partial}{\partial x} \in \Xi$ on U where $u_\alpha, f \in C^\infty(U)$ and $\frac{\partial u_\alpha}{\partial x} = \frac{\partial g_{\alpha\beta}}{\partial x} = 0$. Local coordinates of this form will be called *Walker coordinates*.

For any given Walker coordinates an integrable realization of the screen bundle is given by $S := \mathrm{span}\{\frac{\partial}{\partial y^\alpha} : 1 \le \alpha \le n\}$. In this case $\Xi = \mathrm{span}\{\frac{\partial}{\partial x}\}$ and $\Theta = \mathrm{span}\{Z\}$ with $Z := \frac{1}{2}(g^{\alpha\beta}u_\alpha u_\beta - f)\frac{\partial}{\partial x} - g^{\alpha\beta}u_\alpha \frac{\partial}{\partial y^\beta} + \frac{\partial}{\partial z}$. The parallel transport equations immediately imply $Hol(M_{xz}, \nabla^{M_{xz}}) \subset Hol(S, \nabla^S)$ where M_{xz} is the Riemannian submanifold given by the $\frac{\partial}{\partial y^\alpha}$-coordinates.[6] Note however, that all indecomposable Lorentzian holonomies have been realized in [Gal06] by Walker coordinates for which $Hol(M_{xz}, \nabla^{M_{xz}}) = 0$.

[5] In fact, we can identify this class with the homomorphism $H_1(X,\mathbb{R}) \twoheadrightarrow pr_{\mathbb{R}\cdot v} \circ Hol(\nabla^E)|_{\mathbb{R}\cdot v} \subset (\mathbb{R}_+, \cdot) \overset{\log}{\twoheadrightarrow} (\mathbb{R}, +)$ which is induced by $\pi_1(X) \twoheadrightarrow Hol_p(\nabla)/Hol^0_p(\nabla)$ as $\mathbb{R}$ is abelian and torsion-free.

[6] The necessary computations for a Walker coordinate neighborhood which are used in this text are summarized in appendix 3.2.

Corollary 2.15. *Let (E, h, ∇^E, π, X) be a pseudo-Riemannian vector bundle of signature $(1, q+1)$ and $q > 0$ such that $\mathfrak{hol}_p^{loc}(\nabla^E)$ is weakly irreducible with index 1 for all $p \in X$. Then $Hol(\nabla^E)$ is weakly irreducible with index 1 and $Hol^0(\nabla^E)$ is of type 2 or 4 if and only if there is a $\mathfrak{hol}_p^{loc}(\nabla^E)$-invariant non-zero vector for all $p \in X$.*

Proof. Since $\mathfrak{hol}_p^{loc}(X, g) = \mathfrak{hol}_p(U_\alpha, g|_{U_\alpha})$ for some neighborhood $U_\alpha \ni p$ we have $\Xi^{U_\alpha}|_{U_\alpha \cap U_\beta} = \Xi^{U_\beta}|_{U_\alpha \cap U_\beta}$, i.e., there is a $Hol(\nabla^E)$-invariant isotropic subbundle $\Xi \subset E$ on X. The second statement follows from Lemma 2.14. □

However, the existence of a covering of X by weakly irreducible Walker coordinates does not imply reducibility of $Hol^0(X, g)$ as we can see from

Example 2.16. *Let $f_1, f_2 \in C^\infty(\mathbb{R})$ such that $f_1|_{]-\infty,-1]} = f_2|_{[1,\infty[} = 1$ and $f_1|_{[-\frac{1}{2},\infty[} = f_2|_{]-\infty,\frac{1}{2}]} = 0$. Let $X = \mathbb{R}^3$ and define*

$$g := 2dxdz + y^2 f_1(z)dz^2 + y^2 f_2(z)dx^2 + dy^2.$$

Then $Hol(X, g) = SO_0(1, 2)$.

Proof. Let $U_1 := \{(x, y, z) \in \mathbb{R}^3 : z < \frac{1}{2}\}$ and $U_2 := \{(x, y, z) \in \mathbb{R}^3 : z > -\frac{1}{2}\}$. If $p_1 := (0, 0, -2)$ and $p_2 := (0, 0, +2)$ then the computations in the appendix imply $R_{p_1}(\partial_z, \partial_y)\partial_y = -\nabla_{\partial_y}(y\partial_x) = -\partial_x$ and $R_{p_2}(\partial_x, \partial_y)\partial_y = -\partial_z$. Hence, $Hol_{p_1}(U_1, g|_{U_1})$ and $Hol_{p_2}(U_2, g|_{U_2})$ are weakly irreducible. Consider the curve $\gamma : [0, 1] \to \mathbb{R}^3$ where $\gamma(t) := (0, 0, 2(2t-1))$. For any vector $V_0 \in T_{p_1}X$ its parallel displacement V_t along γ is the solution to $0 = \dot{V}^k + \Gamma_{ij}^k \dot{\gamma}^i V^j = \dot{V}_t^k + 4t\Gamma_{zj}^k V^j$. Using the formulas in 3.2 we derive $\Gamma_{zj}^k|_{\gamma(t)} = 0$. Hence, $\partial_x, \partial_y, \partial_z$ are parallel along γ and we conclude $(\tau_\gamma^{-1} \circ R_{p_2}(\partial_x, \partial_y) \circ \tau_\gamma)(\partial_y) = -\partial_z$. Therefore, the endomorphism $\tau_\gamma^{-1} \circ R_{p_2}(\partial_x, \partial_y) \circ \tau_\gamma$ is not contained in $\mathfrak{hol}_{p_1}(U_1, g|_{U_1})$. However, the Ambrose-Singer theorem implies $\tau_\gamma^{-1} \circ R_{p_2}(\partial_x, \partial_y) \circ \tau_\gamma \in \mathfrak{hol}_{p_1}(X, g)$, i.e., $Hol(X, g) = SO_0(1, 2)$. □

2.2 The Total Space of a Circle Bundle as a Lorentzian Manifold

In this section we introduce a class of Lorentzian metrics on the total space of an S^1-bundle over a manifold admitting a nowhere vanishing closed 1-form for which the vertical vector field will be recurrent or parallel. In the subsequent section we will apply foliation theory to identify this class as the “prototype" of weakly irreducible Lorentzian manifolds with compact leaves. Moreover, this construction allows to show optimality of the Bochner technique for decent spacetimes.

In the following we write $[\frac{\psi}{2\pi}] \in H^2(M, \mathbb{Z})$ if ψ is a closed 2-form on the manifold M and $[\frac{\psi}{2\pi}] \in \mathrm{Im}(H^2(M, \mathbb{Z}) \to H^2_{dR}(M, \mathbb{R}))$. We state the main construction method.

Proposition 2.17. *Let (M, g) be an $(n+1)$-dimensional Riemannian manifold and η a nowhere vanishing closed 1-form on M. Let ψ be a 2-form on M with $[\frac{\psi}{2\pi}] \in H^2(M, \mathbb{Z})$. Then there exists an S^1-bundle $\pi : X \to M$ satisfying $c_1(X \to M) = -[\frac{\psi}{2\pi}]$ and*

1. *There is a global nowhere vanishing 1-form θ on X such that*

$$\tilde{g}_f := 2\theta\pi^*\eta + f \cdot \pi^*\eta^2 + \pi^* g$$

defines a Lorentzian metric on X for any $f \in C^\infty(X)$.

2. *Given $p \in X$ and a local 1-form ϕ with $\psi = d\phi$ there are local coordinates $(x, y^1, \dots, y^n, z)$ around p such that*

$$\tilde{g}_f = 2dxdz + 2(u_i + g_{i(n+1)})dy^i dz + (f + 2u_{n+1} + g_{(n+1)(n+1)})dz^2 + g_{ij}dy^i dy^j,$$

where $\phi = u_i dy^i + u_{n+1}dz$.

3. *The $U(1)$-action of $X \to M$ acts by isometries on $(X, \tilde{g}_f)$ if f is constant on the fibers.*

4. *The vertical vector field is a global lightlike vector field, which is parallel if and only if f is constant on the fibers.*

Proof. Consider the smooth complex line bundle $L \to M$ given by $c_1^{-1}(-[\frac{\psi}{2\pi}])$ and some Hermitian metric h on L. The curvature endomorphism of a Hermitian connection ∇_h on (L, h) is given by a closed imaginary 2-form $\sqrt{-1}F_h$ and $[\sqrt{-1}F_h] = -2\pi\sqrt{-1}c_1(L)$.[7] Hence, $F_h - \psi = d\lambda$ for some real 1-form λ and $\nabla_L := \nabla_h - i\lambda$ is another Hermitian connection on (L, h) whose curvature endomorphism is given by $\sqrt{-1}\psi$. The metric h provides a $U(1)$-reduction of the $GL(1, \mathbb{C})$-bundle L and since ∇_L is Hermitian it reduces as well. Thus, we derive an S^1-bundle $X := \{v \in L : h(v, v) = 1\} \to M$ together with the $U(1)$-connection ∇_L.

Consider the 1-form η on M. By Frobenius' theorem we can find for all $q \in M$ local coordinates $(y_1, \dots, y_n, z)$ on some neighborhood $U \ni q$ such that $\eta = dz$. Moreover, we may assume that $X \to M$ is trivial over U and $\psi = d\phi_U$. Let $s_U : U \to L$ be a unit length section such that[8]

$$\nabla_L s_U = \sqrt{-1}\phi_U \otimes s_U.$$

Using s_U and the S^1-action on X we define local coordinates

$$(x^0, \dots, x^{n+1}) := (x, y^1, \dots, y^n, z)$$

around $p := s_U(q)$ given by $e^{\sqrt{-1}x}s_U(y_1, \dots, y_n, z)$.

In order to construct the 1-form θ we consider another coordinate neighborhood $V \subset M$. Suppose there is a 1-form ϕ_V on V such that $d\phi_V = \psi$ and a unit length section

[7] The curvature endomorphism of any Hermitian connection on a complex vector bundle E is a 2-form with values in the unitary Lie algebra $\mathfrak{u}(\dim_{\mathbb{C}} E)$ and here we identify $\mathfrak{u}(1) \cong \sqrt{-1} \cdot \mathbb{R}$. Remember, that $c_1(L) = \frac{\sqrt{-1}}{2\pi}[\sqrt{-1}F_h] = \pm\delta$ where δ is the boundary operator of the exponential sequence. The sign in this equation depends on various conventions (cf. [Huy05, Prop. 4.4.12]).

[8] If $t : U \to L$ is any unit length section we have $\nabla_L t = \sqrt{-1}\alpha \otimes t$ for some real 1-form α. Hence, $\alpha - \phi_U = df$ for some $f \in C^\infty(U)$ and $s_U := e^{-\sqrt{-1}f}t$ has the local connection form $\sqrt{-1}\phi_U$.

$s_V : V \to L$ such that $\nabla_L s_V = \sqrt{-1}\phi_V \otimes s_V$. If $U \cap V \neq \emptyset$ we have $s_V = e^{\sqrt{-1}g_{UV}} s_U$. Therefore,

$$\begin{aligned}\nabla_L s_V &= \nabla_L(e^{\sqrt{-1}g_{UV}} s_U) = \sqrt{-1}dg_{UV} \otimes s_V + \sqrt{-1}\phi_U \otimes s_V \\ &= \sqrt{-1}(dg_{UV} + \phi_U)s_V\end{aligned}$$

and we conclude $dg_{UV} = \phi_V - \phi_U$. Given the local coordinates defined by s_U and s_V we observe

$$e^{\sqrt{-1}x_V} s_V(q) = e^{\sqrt{-1}x_V} e^{\sqrt{-1}g_{UV}} s_U(q) = e^{\sqrt{-1}(x_U+c)} s_U(q)$$

for $q \in U \cap V$ and some $c \in \mathbb{R}$, i.e., $dx_U - dx_V = dg_{UV} = \phi_V - \phi_U$. Hence, $dx_U + \phi_U$ glues to a global nowhere vanishing 1-form θ on X and $d\theta = \pi^*\psi$.

In order to show that $\tilde{g}_f := 2\theta\pi^*\eta + f \cdot \pi^*\eta^2 + \pi^*g$ is a Lorentzian metric we apply the local coordinate expression

$$\tilde{g}_f = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & g_{11} & \cdots & g_{1n} & g_{1(n+1)} + u_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & g_{n1} & \cdots & g_{nn} & g_{n(n+1)} + u_n \\ 1 & g_{(n+1)1} + u_1 & \cdots & g_{(n+1)n} + u_n & f + u_{n+1} + g_{(n+1)(n+1)} \end{pmatrix}$$

and conclude $\det(\tilde{g}_{ij}) < 0$ since $(g_{\alpha\beta})_{1 \leq \alpha,\beta \leq n}$ is the Riemannian metric g restricted to the submanifold $\{(y^1, \ldots, y^n, \text{const.})\}$.

If $f \in C^\infty(X)$ is constant on the fibers the $U(1)$-action of the bundle leaves $\tilde{g}_f$ invariant since $\sqrt{-1}\theta$ is the connection 1-form of ∇_L and all other terms of $\tilde{g}_f$ are pullbacks. By definition of $\tilde{g}_f$ the vertical vector field is lightlike and the coordinate expression of $\tilde{g}_f$ implies (cf. Appx. 3.2) for $i, k \in \{0, \ldots, n+1\}$

$$\Gamma^k_{0i} = \frac{1}{2}\delta_{i(n+1)}\delta_{k0}\frac{\partial(f + u_{n+1} + g_{(n+1)(n+1)})}{\partial x} = \frac{1}{2}\delta_{i(n+1)}\delta_{k0}\frac{\partial f}{\partial x}.$$

Therefore, the vertical vector field is parallel if and only if f is constant on the fibers. □

Remark 2.18. Up to diffeomorphism X depends only on the choice of the class $[\frac{\psi}{2\pi}] \in H^2(M, \mathbb{Z})$. However, the Lorentzian metric $\tilde{g}_f$ depends on the representative $\psi \in [\psi] \in H^2(M, \mathbb{R})$. This will be crucial once we study the screen holonomy of $(X, \tilde{g}_f)$. □

The topology of $X \to M$ can be studied using the following well known facts.

Remark 2.19. Suppose $(X, \tilde{g}_f)$ is constructed as in Prop. 2.17. Then

- $\to H^i(X) \to H^{i-1}(M) \xrightarrow{c_1\wedge\cdot} H^{i+1}(M) \xrightarrow{\pi^*} H^{i+1}(X) \to$ (Gysin sequence)
- $\to \pi_i(S^1) \to \pi_i(X) \to \pi_i(M) \to \pi_{i-1}(S^1) \to$ (Serre homotopy sequence)

□

In the following a weakly irreducible Lorentzian manifold (X, g) is said to be of type $\alpha \in \{1, \ldots, 4\}$ if there is $p \in X$ such that $\mathfrak{hol}_p(X, g)$ is of type α in Thm. 2.11.

Proposition 2.20. *Suppose $(X, \tilde{g}_f) \to (M, g)$ is constructed as in Prop. 2.17 such that $(X, \tilde{g}_f)$ is weakly irreducible. Then $(X, \tilde{g}_f)$ is of type 1 if and only if $\frac{\partial f}{\partial x}|_p \neq 0$ for some $p \in X$. In particular, $(X, \tilde{g}_f)$ is not of type 3.*

Proof. Consider the vertical line subbundle Ξ of TX spanned by $\frac{\partial}{\partial x}$ and its induced connection ∇^Ξ. Write R^{∇^Ξ} for the curvature of ∇^Ξ. Given the local coordinates $(x^0, x^1, \ldots, x^n, x^{n+1}) := (x, y^1, \ldots, y^n, z)$ from Prop. 2.17 we know

$$\tilde{g}_f = 2dxdz + 2\tilde{u}_i dy^i dz + \tilde{f}dz^2 + g_{ij}dy^i dy^j.$$

Using $\Gamma^k_{0i} = \frac{1}{2}\delta_{i(n+1)}\delta_{k0}\frac{\partial \tilde{f}}{\partial x^0}$ we compute

$$R^{\nabla^\Xi}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^0} = \frac{1}{2}\left(\delta_{j(n+1)}\frac{\partial^2 \tilde{f}}{\partial x^i \partial x^0} - \delta_{i(n+1)}\frac{\partial^2 \tilde{f}}{\partial x^j \partial x^0}\right)\frac{\partial}{\partial x^0}.$$

It is shown in [Bez05, Prop. 6.1] that $\mathfrak{hol}(X, \tilde{g}_f)$ is of type 2 or 4 if and only if $R^{\nabla^\Xi} = 0$. Using the formula for R^{∇^Ξ} we conclude that $(X, \tilde{g}_f)$ is not of type 1 or 3 if $\frac{\partial \tilde{f}}{\partial x^0} = 0$. Suppose $0 = R^{\nabla^\Xi}(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^{n+1}})\frac{\partial}{\partial x^0} = \frac{1}{2}\frac{\partial^2 \tilde{f}}{\partial (x^0)^2}$. If $\tilde{f}$ is restricted to a fiber S^1 then we conclude that $\tilde{f}$ is constant on the fiber, i.e., $\frac{\partial \tilde{f}}{\partial x^0} = 0$. In particular, $R^{\nabla^\Xi} = 0$ if and only if $\frac{\partial \tilde{f}}{\partial x^0} = 0$.

Hence, $(X, \tilde{g}_f)$ is of type 1 or 3 if and only if $\frac{\partial \tilde{f}}{\partial x^0}|_p \neq 0$ for some $p \in X$. For the last statement we assume $(X, \tilde{g}_f)$ is of type 3. In this case [Bez05, Prop. 6.2] implies $R^{\nabla^\Xi}(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^{n+1}}) = 0$ and therefore $R^{\nabla^\Xi} = 0$. This is a contradiction. □

Definition 2.21. *If $(X, \tilde{g}_f)$ is constructed as in Prop. 2.17 then we say $f \in C^\infty(X)$ is suitable if $\mathfrak{hol}(X, \tilde{g}_f)$ is weakly irreducible and not of type 4.*

Let us explain how suitable functions can be constructed on X. By Prop. 2.17 we have a local coordinate neighborhood $U \subset X$ around some $p \in X$ such that $x(p) = y^\alpha(p) = 0$ and

$$\tilde{g}_f = 2dxdz + 2(u_i + g_{i(n+1)})dy^i dz + (f + 2u_{n+1} + g_{(n+1)(n+1)})dz^2 + g_{ij}dy^i dy^j$$

for any choice of $f \in C^\infty(X)$. Construct $\tilde{f} \in C^\infty(X)$ as follows. On $p \in V \subset U$ we define $\tilde{f} := f - 2u_{n+1} - g_{(n+1)(n+1)}$ where f is one of the functions in Lemma 4.3. If f does not depend on x then $\tilde{f}$ is a function on $\check{V} \subset M$ and we can extend $\tilde{f}$ to a function $\tilde{f} \in C^\infty(M)$ using a partition of unity on M. In this case, $(\tilde{f} \circ \pi) \in C^\infty(X)$.

Otherwise, we extend $\tilde{f}$ using a partition of unity on X. If $\mathfrak{h} := \mathfrak{hol}_p(X, \tilde{g}_{\tilde{f}})$ then Lemma 4.2 implies that $\mathfrak{h}$ is weakly irreducible and not of type 4 if its orthogonal part $\mathfrak{g} \subset \mathfrak{so}(\dim X - 2)$ is trivial or irreducible.

In all constructions to follow the screen holonomy $\mathfrak{g}$ (cf. Remark 2.8) will be either irreducible or trivial. Hence, we will assume w.l.o.g. that $f \in C^\infty(X)$ is suitable.

Definition 2.22. *Let (N, g) be a Riemannian manifold with $[\frac{\psi}{2\pi}] \in H^2(N, \mathbb{Z})$ and L a 1-dimensional manifold. If $\eta = dz$ is the coordinate 1-form*[9] *on L and $\tilde{X}$ is the S^1-bundle corresponding to $-[\frac{\psi}{2\pi}]$ let $X = \tilde{X} \times L$ and*

$$\tilde{g}_f := 2\theta\pi^*\eta + f \cdot \pi^*\eta^2 + \pi^*g$$

for some $f \in C^\infty(X)$. Then we say $(X, \tilde{g}_f)$ is of toric type over (N, g).[10].

If we consider $(M = N \times L, g + \eta^2)$ then $(X, \tilde{g}_f)$ is a special case of Prop. 2.17 once we replace f there by $f - 1$. Recall that $\partial_z u_\alpha = \partial_z g_{\alpha\beta} = 0$ for all $\alpha, \beta \in \{1, \ldots, n\}$ and $u_{n+1} = 0$ if $(X, \tilde{g}_f)$ is of toric type.

Proposition 2.23. *Let $(X, \tilde{g}_f)$ be of toric type and $\pi : \tilde{X} \to M$ the corresponding S^1-bundle. Then:*

- *The horizontal distribution in TX is isomorphic to the screen bundle,*
- *$Hol(M, g) \subset G := Hol(\nabla^{\mathcal{S}})$,*
- *$\mathfrak{hol}^{loc}_{\pi(p)}(M, g) \subset \mathfrak{hol}^{loc}_{(p,q)}(\nabla^{\mathcal{S}})$.*

Proof.

1. We have $X = \tilde{X} \times L$ with $\dim L = 1$. If $\frac{\partial}{\partial z}$ is the coordinate vector field on L and V is the vertical vector field of $\tilde{X}$ then we derive a non-canonical realization $S := \operatorname{span}\{V, Z := -\frac{1}{2} f V + \frac{\partial}{\partial z}\}^\perp$ of the screen bundle. In particular, using $\tilde{g}_f = 2dxdz + 2u_i dy^i dz + g_{ij} dy^i dy^j + f dz^2$ we observe

$$\tilde{g}_f(V, \frac{\partial}{\partial y^i} - u_i \frac{\partial}{\partial x}) = \tilde{g}_f(Z, \frac{\partial}{\partial y^i} - u_i \frac{\partial}{\partial x}) = 0,$$

   i.e., $Y_i := \frac{\partial}{\partial y^i} - u_i \frac{\partial}{\partial x} \in S$. The horizontal distribution in $T\tilde{X}$ is given by $\operatorname{Ker}(\theta)$ and its pullback H to X is the horizontal distribution in TX. Since $\tilde{g}_f := 2\theta\eta + f \cdot \pi^*\eta^2 + \pi^*g$ and $\eta(V) = 0$ as well as $\theta(V) = 1$ we conclude $H \perp V$ and $H \perp \frac{\partial}{\partial z}$, i.e., $H = S$.

2. Fix $(p, q) \in \tilde{X} \times L$ and let $x := \pi(p)$. For each $a \in Hol_x(M, g)$ we construct a loop $\tilde{\gamma} : I \to X$ on which the $\nabla^{\tilde{g}_f}$-parallel displacement induces $a \in G$. More precisely, let $\gamma : [0, 1] \to M$ be a loop such that $\gamma(0) = x$ and let $\tilde{\delta} : [0, 1] \to \tilde{X}$ be the horizontal lift of γ such that $\tilde{\delta}(0) = p$. Define $u := \tilde{\delta}(1)$. If $u \neq p$ let $\tilde{\beta}$ be the integral curve of the vertical vector field in the fiber $\pi^{-1}(x)$ connecting u and p. We define

$$\tilde{\gamma} := \begin{cases} (\tilde{\delta} * \tilde{\beta}, q) & \text{if } u \neq p, \\ (\tilde{\delta}, q) & \text{otherwise.} \end{cases}$$

[9]If $L = S^1$ we define dz using the coordinates $(0, 1) \to S^1$ such that $z \mapsto e^{2\pi\sqrt{-1}z}$.

[10]If $(X, \tilde{g}_f)$ is of toric type and $L = S^1$ then X is a torus bundle over N where one direction in the fibers is trivial.

Let $v \in T_xM$ and let v_t be its (M,g)-parallel displacement along γ. Write $\tilde{v}_t$ for the horizontal lift of v_t to $\tilde{\gamma}$. First, we consider the $\nabla^{\tilde{g}_f}$-parallel displacement w_t of $\tilde{v}_0$ along $\delta = (\tilde{\delta}, q) : I \to X$ and show $\tilde{v}_t = pr_S(w_t)$.

The set $J \subset I$ on which this equation holds is non-empty and closed. In order to show that $J \subset I$ is open we may use local coordinates. Let $1 \leq \alpha, \beta, k \leq n$. Since $\Xi^\perp$ is $\nabla^{\tilde{g}_f}$-parallel we have $w_t^{n+1} = 0$. Moreover, $\dot{\delta}_t^{n+1} = 0$ since δ is a curve tangent to $\Xi^\perp$. The computations in Appx. 3.2 imply $\Gamma^k_{\cdot 0} = 0$ and

$$0 = \dot{w}_t^k + \Gamma^k_{ij}\dot{\delta}_t^i w_t^j = \dot{w}_t^k + \Gamma^k_{\alpha\beta}\dot{\gamma}_t^\alpha w_t^\beta = \dot{w}_t^k + \tilde{\Gamma}^k_{\alpha\beta}\dot{\gamma}_t^\alpha w_t^\beta,$$

where $\tilde{\Gamma}^k_{\alpha\beta}$ are the Christoffel symbols of (M,g). This shows $w_t^k = v_t^k$, i.e., $pr_S(w_t) = \tilde{v}_t$.

Suppose $u \neq p$ and consider the $\nabla^{\tilde{g}_f}$-parallel displacement $\tilde{v}_t$ of a vector $\tilde{v} \in \Xi_u^\perp$ along $\beta = (\tilde{\beta}, q)$. Again we can work in a local coordinate chart. As $\tilde{\beta}$ is an integral curve of the vertical vector field we compute

$$0 = \dot{\tilde{v}}_t^\ell + \Gamma^\ell_{ij}\dot{\beta}_t^i \tilde{v}_t^j = \dot{\tilde{v}}_t^\ell + \Gamma^\ell_{0j}\dot{\beta}_t^0 \tilde{v}_t^j = \dot{\tilde{v}}_t^\ell + \Gamma^\ell_{0(n+1)}\dot{\beta}_t^0 \underbrace{\tilde{v}_t^{n+1}}_{=0} = \dot{\tilde{v}}_t^\ell,$$

i.e., $\tilde{v}_t^k = \text{const}$. Hence, the $\nabla^{\tilde{g}_f}$-parallel displacement along $\tilde{\gamma}$ induces $a \in G$.

3. For the last statement we may use the same arguments as above.

$\square$

Example 2.24. *Let (M,g) be a Riemannian manifold such that $\mathfrak{hol}(M,g) = \mathfrak{so}(n)$. If $(X, \tilde{g}_f)$ is of toric type over (M,g) then*

$$\mathfrak{hol}(X, \tilde{g}_f) = \begin{cases} \mathfrak{so}(n) \ltimes \mathbb{R}^n & \text{if } \frac{\partial f}{\partial x} \equiv 0, \\ (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n & \text{otherwise}, \end{cases}$$

if $f \in C^\infty(X)$ is a suitable function. $\square$

In order to analyze the screen holonomy of toric type Lorentzian manifolds we fix the non-canonical realization $S := \text{span}\{V, Z := \frac{\partial}{\partial z} - \frac{1}{2}fV\}^\perp$ of the screen bundle. Hence, S is locally generated by $Y_i := \partial_i - u_i\partial_0$ for some local coordinate frame $\frac{\partial}{\partial y^i}$ on M. Using the computations in Appx. 3.2 we conclude $\nabla^{\tilde{g}_f}_{\partial_i} Y_j = (\Gamma^k_{ij} - u_j\Gamma^k_{0i})\partial_k - \frac{\partial u_j}{\partial x^i}\partial_0$ implying $\nabla^{\tilde{g}_f}_{\partial_0} Y_j = -\frac{\partial u_j}{\partial x}\partial_0 = 0$ and

$$\nabla^S_Z Y_\alpha = \frac{1}{2}g^{\gamma\beta}(\partial_\alpha u_\beta - \partial_\beta u_\alpha)Y_\gamma = \frac{1}{2}g^{\gamma\beta}(\psi(\partial_\alpha, \partial_\beta))Y_\gamma.$$

Moreover, for $\alpha \in \{1, \ldots, n\}$ we have $\nabla^S_{Y_i} Y_j = pr_S(\nabla^{\tilde{g}_f}_{\partial_i} Y_j) = \Gamma^\alpha_{ij} Y_\alpha$, i.e.,

$$R^{\nabla^S}(Y_i, Y_j)Y_k = pr_S(R^{\nabla^{\tilde{g}_f}}(\partial_i, \partial_j)\partial_k) = pr_S(R^{(M,g)}(\partial_i, \partial_j)\partial_k)$$

and $R^{\nabla^S}(\partial_0, Y_i)Y_j = R^{\nabla^S}(\partial_0, Z)Y_k = 0$. In particular, we have the identity

$$Ric^S(Y_i, Y_j) := \sum_{\alpha=1}^{\dim M} \tilde{g}_f(R^{\nabla^S}(Y_i, Y_\alpha)Y_\alpha, Y_j) = Ric^{(M,g)}(\partial_i, \partial_j),$$

which will be applied in the next section.

Given Prop. 2.23 we have a lower bound for the full screen holonomy $G := Hol(\nabla^S)$. In order to derive an upper bound for G we construct a tensor $\tilde{T}$ on S such that $\nabla^S_\cdot \tilde{T} = 0$, i.e., $G \subset \mathrm{Stab}_{O(n)}\{\tilde{T}\}$. More precisely, given a (M, g)-parallel tensor T on (M, g) we consider its lift to the horizontal bundle of $\tilde{X} \to M$. Using Prop. 2.23 the trivial extension of the lift along the z-direction provides a tensor $\tilde{T}$ on S. In this case, the construction implies $\nabla^S_{Y_i}\tilde{T} = \nabla^S_{\partial_0}\tilde{T} = 0$. Hence, we have to consider the condition $\nabla^S_Z \tilde{T} = 0$ which will generally impose a restriction on the Chern class of $\tilde{X} \to M$.

Proposition 2.25. *Let (M, g) be a Riemannian manifold such that $[\frac{\psi}{2\pi}] \in H^2(M, \mathbb{Z})$. Suppose $(X = \tilde{X} \times L, \tilde{g}_f)$ is of toric type where $\tilde{X} \to M$ is the S^1-bundle corresponding to $-[\frac{\psi}{2\pi}]$ and $\tilde{g}_f$ is defined using the representative $\psi \in [\psi]$ with $f \in C^\infty(X)$ suitable.*

1. *$Hol^0(\nabla^S) = 0 \Leftrightarrow (M, g)$ is flat and $\nabla^{(M,g)}\psi = 0$.*
2. *If (M, J, g) is Kähler then $\nabla^S \tilde{J} = 0 \Leftrightarrow \psi \in \mathcal{A}^{1,1}(M, J)$.*
3. *If (M, J, g) is Kähler with a parallel holomorphic volume form Ω then $\nabla^S \tilde{J} = \nabla^S \tilde{\Omega} = 0 \Leftrightarrow \psi \in \mathcal{A}^{1,1}(M, J)$ is a primitive form, i.e., $\Lambda\psi = 0$ where Λ is the dual Lefschetz operator.*
4. *If (M, J_1, J_2, J_3, g) is hyperkähler then $\nabla^S \tilde{J}_1 = \nabla^S \tilde{J}_2 = \nabla^S \tilde{J}_3 = 0 \Leftrightarrow \psi \in \mathcal{A}^{1,1}(M, J_1) \cap \mathcal{A}^{1,1}(M, J_2)$.*
5. *If (M, ϕ, g) is a G_2-manifold then $\nabla^S \tilde{\phi} = 0 \Leftrightarrow \mathcal{BI}(C_{24}(\psi \otimes \phi)) = 0$, where C_{24} is the metric contraction over the second and the fourth slot and $\mathcal{BI}$ is the Bianchi projector.*[11]
6. *If (M, Ω, g) is a $Spin(7)$-manifold then $\nabla^S \tilde{\Omega} = 0 \Leftrightarrow \mathcal{AB}(C_{24}(\psi \otimes \Omega)) = 0$, where $\mathcal{AB}$ is the alternating cyclic sum.*

Proof.

1. If $Hol^0(\nabla^S) = 0$ then Prop. 2.23 implies that (M, g) is flat. Using a local orthonormal coordinate frame on (M, g) such that $\Gamma^\gamma_{\alpha\beta} = 0$ we compute

$$\begin{aligned} R^{\nabla^S}(Y_i, Z)Y_k &= \frac{1}{2}\sum_\alpha \nabla^S_{Y_i}(\psi(\partial_k, \partial_\alpha)Y_\alpha) = \frac{1}{2}\sum_\alpha (Y_i(\psi(\partial_k, \partial_\alpha)))Y_\alpha \\ &= \frac{1}{2}\sum_\alpha ((\nabla^{(M,g)}_{\partial_i}\psi)(\partial_k, \partial_\alpha))Y_\alpha, \end{aligned}$$

[11] We do not introduce the notion of G_2- and $Spin(7)$-manifold since they are not anymore considered in this exposition. Instead, we refer to [Joy00] and to [Ver05] for the existence question of certain parallel forms on G_2-manifolds.

i.e. $R^{\nabla^S} = 0 \Leftrightarrow R^{(M,g)} = 0$ and $\nabla^{(M,g)}\psi = 0$.

2. On the Kähler manifold (M, J, g) we fix local coordinates $(y^1, \dots, y^{2m})$ such that $\partial_{k+m} = J\partial_k$. As usual we write Y_i for the extended horizontal lift of ∂_j. Hence, $Y_{k+m} = \tilde{J}Y_k$. Since $\nabla^{(M,g)}J = 0$ the only non-vanishing $\nabla^S_{\cdot}\tilde{J}$ can be $\nabla^S_Z\tilde{J}$. Using the expression for $\nabla^S_Z Y_\alpha$ we compute

$$\tilde{g}_f(\nabla^S_Z(\tilde{J}Y_j), Y_\ell) = \frac{1}{2}g^{k\alpha}\psi(J\partial_j, \partial_\alpha)\tilde{g}_f(Y_k, Y_\ell) = \frac{1}{2}\psi(J\partial_j, \partial_\ell)$$

and

$$\begin{aligned}\tilde{g}_f(\tilde{J}(\nabla^S_Z Y_j), Y_\ell) &= -\tilde{g}_f(\nabla^S_Z Y_j, \tilde{J}Y_\ell) = -\frac{1}{2}g^{k\alpha}\psi(\partial_j, \partial_\alpha)\tilde{g}_f(Y_k, \tilde{J}Y_\ell)\\ &= -\frac{1}{2}\psi(\partial_j, J\partial_\ell),\end{aligned}$$

i.e., $\nabla^S_Z\tilde{J} = 0 \Leftrightarrow \psi(J\partial_j, \partial_\ell) + \psi(\partial_j, J\partial_\ell) = 0$.

3. If (M, J, g) is a Kähler manifold we fix local coordinates as above and define $Z_k := \frac{1}{2}(Y_k - \sqrt{-1}\tilde{J}Y_k)$ as well as $\frac{\partial}{\partial z^k} := \frac{1}{2}(\partial_k - \sqrt{-1}J\partial_k)$ for $1 \le k \le m := \dim_{\mathbb{C}} M$. If (M, J) admits a parallel holomorphic volume form Ω, i.e., a parallel, nowhere vanishing, $\bar{\partial}$-closed $(m, 0)$-form then $\tilde{\Omega}(Z_1, \dots, Z_m) = \Omega(\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^m})$ and the only non-vanishing term of $\nabla^S_{\cdot}\tilde{\Omega}$ can be $\nabla^S_Z\tilde{\Omega}$. Using $-\sqrt{-1}\tilde{J}Z_k = Z_k$ and $\nabla^S\tilde{J} = 0$ we compute for a fixed $k \in \{1, \dots, m\}$

$$\begin{aligned}\tilde{\Omega}(Z_1, \dots, \nabla^S_Z Z_k, \dots, Z_m) &= \frac{1}{2}g^{\gamma\beta}\psi(\partial_k, \partial_\beta)\tilde{\Omega}(\dots, \frac{1}{2}(Y_\gamma - \sqrt{-1}\tilde{J}Y_\gamma), \dots)\\ &= \frac{1}{2}\sum_{\gamma \le m} g^{\gamma\beta}\psi(\partial_k, \partial_\beta)\tilde{\Omega}(Z_1, \dots, Z_\gamma, \dots, Z_m)\\ &\quad + \frac{1}{2}\sum_{\gamma > m} g^{\gamma\beta}\psi(\partial_k, \partial_\beta)\tilde{\Omega}(Z_1, \dots, \tilde{J}Z_\gamma, \dots, Z_m)\\ &= \frac{1}{2}(g^{k\beta} + \sqrt{-1}g^{k+m\beta})\psi(\partial_k, \partial_\beta)\tilde{\Omega}(Z_1, \dots, Z_m).\end{aligned}$$

Since (M, J, g) is Kähler we may assume $(\partial_1, \dots, \partial_{2m})$ to be orthonormal at some $p \in M$. Since $\partial_{k+m} = J\partial_k$ we compute at $p \in M$

$$\begin{aligned}(\nabla^S_Z\tilde{\Omega})(Z_1, \dots, Z_m) &= -\sum_{k=1}^m \tilde{\Omega}(Z_1, \dots, \nabla^S_Z Z_k, \dots, Z_m)\\ &= -\frac{1}{2}\sqrt{-1}\tilde{\Omega}(Z_1, \dots, Z_m)\sum_{k=1}^m \psi(\partial_k, J\partial_k)\\ &= -\frac{1}{2}\sqrt{-1}(\Lambda\psi)\tilde{\Omega}(Z_1, \dots, Z_m),\end{aligned}$$

i.e., if $\nabla^S\tilde{J} = 0$ then $\nabla^S_Z\tilde{\Omega} = 0 \Leftrightarrow \Lambda\psi = 0$.

4. This follows from the second statement.

5. If (M, ϕ, g) is a G_2-manifold with a parallel positive 3-form ϕ then

$$\begin{aligned}(\nabla^S_Z \tilde{\phi})(Y_\alpha, Y_\beta, Y_\gamma) &= -\frac{1}{2} g^{k\ell}(\psi(\partial_\alpha, \partial_\ell)\phi(\partial_k, \partial_\beta, \partial_\gamma) \\ &\qquad + \psi(\partial_\beta, \partial_\ell)\phi(\partial_\alpha, \partial_k, \partial_\gamma) \\ &\qquad + \psi(\partial_\gamma, \partial_\ell)\phi(\partial_\alpha, \partial_\beta, \partial_k)) \\ &= \frac{1}{2} C_{24}(\psi \otimes \phi)(\partial_\alpha, \partial_\beta, \partial_\gamma) \\ &\quad + \frac{1}{2} C_{24}(\psi \otimes \phi)(\partial_\beta, \partial_\gamma, \partial_\alpha) \\ &\quad + \frac{1}{2} C_{24}(\psi \otimes \phi)(\partial_\gamma, \partial_\alpha, \partial_\beta) \\ &= \frac{1}{2} \mathcal{BI}(C_{24}(\psi \otimes \phi))(\partial_\alpha, \partial_\beta, \partial_\gamma).\end{aligned}$$

6. If (M, Ω, g) is a $Spin(7)$-manifold with a parallel admissible 4-form Ω then

$$\begin{aligned}(\nabla^S_Z \tilde{\Omega})(Y_\alpha, Y_\beta, Y_\gamma, Y_\delta) &= \frac{1}{2} C_{24}(\psi \otimes \Omega)(\partial_\alpha, \partial_\beta, \partial_\gamma, \partial_\delta) \\ &\quad - \frac{1}{2} C_{24}(\psi \otimes \Omega)(\partial_\beta, \partial_\gamma, \partial_\delta, \partial_\alpha) \\ &\quad + \frac{1}{2} C_{24}(\psi \otimes \Omega)(\partial_\gamma, \partial_\delta, \partial_\alpha, \partial_\beta) \\ &\quad - \frac{1}{2} C_{24}(\psi \otimes \Omega)(\partial_\delta, \partial_\alpha, \partial_\beta, \partial_\gamma) \\ &= \frac{1}{2} \mathcal{AB}(C_{24}(\psi \otimes \Omega))(\partial_\alpha, \partial_\beta, \partial_\gamma, \partial_\delta).\end{aligned}$$

□

Consider a basis $(v, e_1, \ldots, e_n, z)$ of $\mathbb{R}^{n+2}$ with $\langle v, z \rangle = 1$, $\langle e_i, e_j \rangle = \delta_{ij}$ and $\langle \cdot, \cdot \rangle = 0$ otherwise. Using this basis we define the group $(\mathbb{R}^* \times O(n)) \ltimes \mathbb{R}^n$ using the the matrix group

$$\left\{ \begin{pmatrix} a & -aY^T B & -\frac{a}{2} Y^T Y \\ 0 & B & Y \\ 0 & 0 & \frac{1}{a} \end{pmatrix} : \begin{array}{c} a \in \mathbb{R}^* = (\mathbb{R}_+, \cdot), \\ B \in O(n), \\ Y \in \mathbb{R}^n \end{array} \right\}$$

where $\mathbb{R}^* = (\mathbb{R}_+, \cdot)$.[12] If (X, g) is a time-orientable Lorentzian manifold such that $\mathfrak{hol}(X, g)$ is weakly irreducible with index 1 then Prop. 2.12 and Lemma 2.14i imply that $Hol(X, g) \subset (\mathbb{R}^* \times O(n)) \ltimes \mathbb{R}^n$.

Suppose $(X, \tilde{g}_f)$ is of toric type such that $f \in C^\infty(X)$ is suitable and $G = Hol(\nabla^S)$. Using Prop. 2.20 we conclude that the recurrent vertical vector field V is $\nabla^{\tilde{g}_f}$-parallel

[12] Note that $((\mathbb{R} \setminus \{0\}, \cdot) \times O(n)) \ltimes \mathbb{R}^n$ is the stabilizer of v in $O(1, n+1)$.

if and only if $\frac{\partial f}{\partial x} \equiv 0$. Hence, the full holonomy group of $(X, \tilde{g}_f)$ is given by

$$Hol(X, \tilde{g}_f) = \begin{cases} G \ltimes \mathbb{R}^{\dim X - 2} & \text{if } \frac{\partial f}{\partial x} \equiv 0, \\ (\mathbb{R}^* \times G) \ltimes \mathbb{R}^{\dim X - 2} & \text{otherwise.} \end{cases}$$

We will prove in Cor. 2.71 that a toric type manifold $(X, \tilde{g}_f)$ over a compact simply connected manifold M does not admit an integrable realization of the screen bundle if $c_1(\tilde{X} \to M) \neq 0$. On the other hand, using [Bau09, Ex. 5.5] we have

Example 2.26. *Suppose $(X, \tilde{g}_f)$ is of toric type over (M, g) for some suitable function $f \in C^\infty(X)$ such that $\psi \equiv 0$ and $\frac{\partial f}{\partial x} \equiv 0$. Then $X = S^1 \times S^1 \times M$ and $\tilde{g}_f = 2dxdz + fdz^2 + g$. Moreover, $Hol(X, \tilde{g}_f) = Hol(M, g) \ltimes \mathbb{R}^{\dim M}$.* □

Let (M, J, g) be a compact Kähler manifold and $(X, \tilde{g}_f)$ as above. By Prop. 2.25 we have $Hol(\nabla^S) \subset U(\frac{\dim_{\mathbb{R}} M}{2})$ if $\psi \in \mathcal{A}^{1,1}(M, J)$. Since $[\frac{\psi}{2\pi}] \in H^2(M, \mathbb{Z})$ the Lefschetz theorem on $(1,1)$-classes (Thm. 1.18) implies $Hol(\nabla^S) \subset U(\frac{\dim_{\mathbb{R}} M}{2})$ if $[\frac{\psi}{2\pi}] \in NS(M, J)$.

On the complex projective space $\mathbb{CP}^n = \bigcup_{i=0}^n U_i$ the tautological line bundle is holomorphic line bundle $\mathcal{O}(-1)$ given by $\{(\ell, x) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : x \in \ell\} \xrightarrow{pr_1} \mathbb{CP}^n$ and if $(z_0 : \ldots : z_n) \mapsto (\frac{z_0}{z_i}, \ldots, \frac{z_n}{z_i})$ are the standard coordinates on U_i the Fubini-Study metric g_{FS} is locally defined by its Kähler form $\omega_{U_i} := \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(\sum_{k=0}^n |\frac{z_k}{z_i}|^2)$. Moreover, the first Chern class $c_1(\mathcal{O}(-1)) \in Im(H^2(\mathbb{CP}^n, \mathbb{Z}) \to H^2(\mathbb{CP}^n, \mathbb{R}))$ (as defined by Chern-Weil theory) is given by $-[\omega_{FS}]$. Moreover, $[\omega_{FS}]$ is a generator of $NS(\mathbb{CP}^n)$.[13]

Example 2.27. *For any $k \in \mathbb{Z}$ let $(X, \tilde{g}_f)$ be of toric type over $(\mathbb{CP}^n, g_{FS})$ such that $c_1(\tilde{X} \to \mathbb{CP}^n) = k[\omega_{FS}]$. Suppose $\tilde{g}_f$ is defined using the representative $-2\pi\omega_{FS} \in -2\pi[\omega_{FS}]$ with $f \in C^\infty(X)$ suitable. Then*

$$Hol(X, \tilde{g}_f) = \begin{cases} U(n) \ltimes \mathbb{R}^{2n} & \textit{if } \frac{\partial f}{\partial x} \equiv 0, \\ (\mathbb{R}^* \times U(n)) \ltimes \mathbb{R}^{2n} & \textit{otherwise.} \end{cases}$$

Proof. Since $Hol(\mathbb{CP}^n, g_{FS}) = U(n)$ Prop. 2.23 implies $U(n) \subset Hol(\nabla^S)$ and we conclude $k[\omega_{FS}] \in H^{1,1}(\mathbb{CP}^n, \mathbb{Z})$ as ω_{FS} is the Kähler form on $\mathbb{CP}^n$. □

Definition 2.28. *Let X be a compact Kähler manifold. A Kähler class $[\omega] \in H^{1,1}(X, \mathbb{Z})$ is a Hodge class and the pair $(X, [\omega])$ is a polarized manifold.*

Kodaira's embedding theorem states that there is a holomorphic embedding $X \hookrightarrow \mathbb{CP}^N$ if and only if X admits a Hodge class.

As the previous example appears to be very restrictive we give a more general construction in the non-symmetric case.

Proposition 2.29. *Let (M^{2n}, J) be a compact simply-connected Kähler manifold such that $-c_1(M, J)$ is a Hodge class, i.e., $c_1(M, J) < 0$.[14] For any Hodge class $[\frac{\psi}{2\pi}] \in$*

[13]For the proofs of these statements we refer to [Huy05, Ex. 4.3.12] and [Kob87, Ch. II].
[14]Explicit examples can be found in [Yau77].

$H^{1,1}((M,J),\mathbb{Z})$ let $(X,\tilde{g}_f)$ be of toric type over (M,J,g) where g is the Einstein-Kähler metric of (M,J) and $c_1(\tilde{X}\to M) = -[\frac{\psi}{2\pi}]$. Suppose $\tilde{g}_f$ is defined using the g-harmonic representative of $[\psi]\in H^2(M,\mathbb{C})$ with $f\in C^\infty(X)$ suitable. Then w.l.o.g.

$$Hol(X,\tilde{g}_f) = \begin{cases} U(n)\ltimes\mathbb{R}^{2n} & \text{if } \frac{\partial f}{\partial x}\equiv 0,\\ (\mathbb{R}^*\times U(n))\ltimes\mathbb{R}^{2n} & \text{otherwise.}\end{cases}$$

Proof. By the Aubin-Yau theorem [Huy05, 4.B.24] there is an Einstein-Kähler metric g on (M,J) which is unique up to homothety. W.l.o.g. we may assume that (M,J,g) is irreducible. Moreover, (M,g) being compact and simply-connected with negative Einstein constant is not symmetric [Bes87, 10.83]. Hence, $Hol(M,J,g) = U(n)$. We conclude the statement since $\psi\in\mathcal{A}^{1,1}(M,J)$ for the harmonic representative ψ of $[\psi]\in H^2(M,\mathbb{C})$.[15] □

Next, we construct Lorentzian manifolds such that $Hol(\nabla^S) = SU(n)$. In the following we say (M,J,g) is a *Calabi-Yau manifold* if M is a compact Kähler manifold such that $Hol(M,g) = SU(n)$. An introduction to Calabi-Yau manifolds can be found in [Joy00]. A Calabi-Yau manifold (M,J,g) admits a parallel holomorphic volume form Ω, i.e., $\Omega\in\mathcal{A}^{p,0}(M,J)$ is nowhere vanishing and $\bar{\partial}$-closed. Given a toric type manifold $(X,\tilde{g}_f)$ over (M,g) consider the associated tensor $\tilde{\Omega}$ on S and let $\check{\Omega} = h\cdot\tilde{\Omega}$ for some nowhere vanishing function $h\in C^\infty(X)$. We say $(\tilde{J},\check{\Omega})$ defines an $SU(n)$-structure on (S,∇^S) if $\nabla^S_\cdot\tilde{J} = \nabla^S_\cdot\check{\Omega} = 0$.

Proposition 2.30. *Let (M,J,g) be a Calabi-Yau manifold and let $(X=\tilde{X}\times L,\tilde{g}_f)$ be of toric type over (M,J,g) where $c_1(\tilde{X}\to M) = -[\frac{\psi}{2\pi}]\in NS(M,J)$ and $\tilde{g}_f$ is constructed using a representative $\psi\in[\psi]$ with $f\in C^\infty(X)$ suitable. Suppose $\Lambda[\psi]\in 4\pi\mathbb{Z}$ or $L=\mathbb{R}$. Then $(\tilde{J}, e^{\frac{\sqrt{-1}}{2}(\Lambda\psi)z}\tilde{\Omega})$ defines an $SU(n)$-structure on (S,∇^S) if and only if ψ is the harmonic representative of $[\psi]$. In this case, we have*

$$Hol(X,\tilde{g}_f) = \begin{cases} SU(n)\ltimes\mathbb{R}^{2n} & \text{if } \frac{\partial f}{\partial x}\equiv 0,\\ (\mathbb{R}^*\times SU(n))\ltimes\mathbb{R}^{2n} & \text{otherwise.}\end{cases}$$

Proof. If $\Lambda[\psi]\in 4\pi\mathbb{Z}$ and $h(z) := e^{\frac{\sqrt{-1}}{2}(\Lambda\psi)z}$ then $h(z) = h(z+1)$, i.e., h defines a function on S^1. Since $\Lambda : H^{1,1}(X)\to\mathbb{C}$ we have $\Lambda\psi = \text{const} = \Lambda[\psi]$ if ψ is the harmonic representative of $[\psi]$ on (M,J,g). The computation in the proof of Prop. 2.25.3 implies $\nabla_Z\tilde{\Omega} = -\frac{\sqrt{-1}}{2}(\Lambda\psi)\tilde{\Omega}$, i.e.,

$$\nabla^S_Z(e^{\frac{\sqrt{-1}}{2}(\Lambda\psi)z}\tilde{\Omega}) = \frac{\sqrt{-1}}{2}(\Lambda\psi)e^{\frac{\sqrt{-1}}{2}(\Lambda\psi)z}\tilde{\Omega} + e^{\frac{\sqrt{-1}}{2}(\Lambda\psi)z}\nabla_Z\tilde{\Omega} = 0.$$

Moreover, if $Y\in\Xi^\perp$ we have $\nabla^S_Y e^{\frac{\sqrt{-1}}{2}(\Lambda\psi)z}\tilde{\Omega} = e^{\frac{\sqrt{-1}}{2}(\Lambda\psi)z}\nabla^S_Y\tilde{\Omega} = 0$ since $\Lambda\psi$ is con-

[15] In order to see this, we consider the degree decomposition $\psi = \psi^{2,0}+\psi^{1,1}+\psi^{0,2}$ and observe $0 = \sum\Delta_{\bar{\partial}}\psi^{i,j}$, i.e., $\Delta_{\bar{\partial}}\psi^{i,j} = 0$ since $\Delta_d\psi^{i,j}\in\mathcal{A}^{i,j}(M,J)$ and the degree decomposition is direct.

stant. Conversely, $\nabla^S_Y e^{\frac{\sqrt{-1}}{2}(\Lambda\psi)z}\tilde{\Omega} = 0$ implies $\Lambda\psi = \text{const}$ and $\nabla^S_Z \tilde{J} = 0$ implies $\psi \in \mathcal{A}^{1,1}(M, J)$. Thus, we have $\bar{\partial}\psi = \partial\psi = 0$ since ψ is a closed $(1,1)$-form. However, the Kähler identities imply $\bar{\partial}^*\psi = \sqrt{-1}[\partial, \Lambda](\psi) = \sqrt{-1}\partial(\Lambda\psi) = 0$, i.e., $\Delta_{\bar{\partial}}\psi = 0$. Finally, we have $SU(n) \subset Hol(\nabla^S)$ by Prop. 2.23 and if $(\tilde{J}, \check{\Omega})$ defines an $SU(n)$-structure on (S, ∇^S) then $Hol(\nabla^S) \subset SU(n)$ (cf. [Joy00, Ch. 6.1]). □

Remark 2.31. Note that the condition $\Lambda[\psi] \in 4\pi\mathbb{Z}$ was imposed to derive a function on $L = S^1$. However, by definition $\Lambda[\psi]$ is constant and once we redefine the toric type metric using an appropriate rescaling $\frac{\partial}{\partial \tilde{z}}$ of the coordinate field $\frac{\partial}{\partial z}$ we can drop the condition $\Lambda[\psi] \in 4\pi\mathbb{Z}$. □

On a complex manifold X we define

$$h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X) \qquad \text{and} \qquad b_k(X) := \dim_{\mathbb{R}} H^k(X, \mathbb{R}).$$

The following simple observation is presumably known, but I could not find an explicit reference. Therefore, we will provide a proof.

Lemma 2.32. *Let $(X, [\omega])$ be a polarized manifold with maximal Picard number, i.e., $\rho(X) = b_2(X) - 2h^{2,0}(X)$. Then $rk(H^{1,1}_{prim}(X, \mathbb{Z})) = \rho(X) - 1$.*

Proof. Since $(X, [\omega])$ is a polarized manifold we can consider the Kähler class $[\omega]$ as an element of $H^2(X, A)$ for $A \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. Since X has maximal Picard number there is a basis $([\omega], c_1, \ldots, c_{\rho(X)-1})$ of $Im(Pic(X) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Q}))$. If $\hat{c}_i$ denotes the image of c_i in $H^2(X, \mathbb{C})$ then the Lefschetz Thm. on $(1,1)$-classes implies $\hat{c}_i \in H^{1,1}(X)$. On $H^{1,1}(X)$ we consider the Hodge-Riemann sesquilinear form

$$\langle \alpha, \beta \rangle := -\int_X [\omega]^{n-2} \wedge \alpha \wedge \bar{\beta},$$

where $n := \dim_{\mathbb{C}} X$. Thus, $\langle \hat{c}_i, \hat{c}_j \rangle, \langle \hat{c}_i, [\omega] \rangle, \langle [\omega], [\omega] \rangle \in \mathbb{Q}$. Moreover, $\langle \cdot, \cdot \rangle$ is positive definite on $H^{1,1}_{prim}(X)$ (cf. [Voi07, Thm. 6.32]) and $\langle [\omega], [\omega] \rangle < 0$. If $\alpha \in H^{1,1}_{prim}(X)$ then $L^{n-1}\alpha = 0$, i.e., $\langle \alpha, [\omega] \rangle = -\int_X [\omega]^{n-1} \wedge \alpha = 0$. Hence, $\langle \cdot, \cdot \rangle$ has signature $(1, h^{1,1} - 1)$.

Using the Gram-Schmidt algorithm we inductively derive a new basis $(v_0, \ldots, v_{\rho(X)-1})$ of $H^{1,1}(X)$ as follows: Define $v_0 := [\omega]$ and let $v_1 := \hat{c}_1 - \frac{\langle \hat{c}_1, [\omega] \rangle}{\langle [\omega], [\omega] \rangle}[\omega]$. However, $\langle \hat{c}_i, [\omega] \rangle, \langle [\omega], [\omega] \rangle \in \mathbb{Q}$ implies $v_0, v_1 \in Im(H^2(X, \mathbb{Q}) \to H^2(X, \mathbb{C})) \cap H^{1,1}(X)$. Suppose we have defined $v_0, \ldots, v_k \in Im(H^2(X, \mathbb{Q}) \to H^2(X, \mathbb{C})) \cap H^{1,1}(X)$ such that $\langle v_i, v_j \rangle = 0$ for $i \neq j$. Then $\mathbb{Q} \ni \langle v_j, v_j \rangle > 0$ for $j \geq 1$ since $\langle [\omega], v_j \rangle = 0$ and we can define $v_{k+1} := \hat{c}_{k+1} - \sum_{j=0}^{k} \frac{\langle \hat{c}_{k+1}, v_j \rangle}{\langle v_j, v_j \rangle} v_j$. Since $\langle \hat{c}_{k+1}, v_j \rangle, \langle v_j, v_j \rangle \in \mathbb{Q}$ we conclude $v_{k+1} \in Im(H^2(X, \mathbb{Q}) \to H^2(X, \mathbb{C})) \cap H^{1,1}(X)$.

Since $\langle [\omega], v_j \rangle = 0$ we derive an orthogonal basis $(v_1, \ldots, v_{\rho(X)-1})$ of $H^{1,1}_{prim}(X)$ and $v_j \in Im(H^2(X, \mathbb{Q}) \to H^2(X, \mathbb{C}))$ implies the statement. □

Given Prop. 2.30 we observe that $(\tilde{J}, \tilde{\Omega})$ itself defines an $SU(n)$ structure on (S, ∇^S) if and only if ψ is the harmonic representative of $[\psi]$ and $[\frac{\psi}{2\pi}] \in H^{1,1}_{prim}((M, J), \mathbb{Z})$.

Moreover, a change of the Kähler class on (M, J) allows us to choose $[\psi] \neq 0$ as we can see from

Corollary 2.33. *Let (M, J, g) be a simply connected Calabi-Yau manifold and suppose $\dim_{\mathbb{C}}(M, J) \geq 3$. Then there is a Kähler metric $\breve{g}$ on (M, J) whose Kähler class is given by $[\breve{\omega}] \in H^2(M, \mathbb{Z})$ such that $Hol(M, \breve{g}) = SU(n)$ and $rk(H^{1,1}_{prim}((M, J, \breve{g}), \mathbb{Z})) = b_2(X) - 1$. In particular, if $(X, \tilde{g}_f)$ is of toric type over $(M, J, \breve{g})$ where $c_1(\tilde{X} \to M) = -[\frac{\psi}{2\pi}] \in H^{1,1}_{prim}((M, J, \breve{g}), \mathbb{Z})$ and $\tilde{g}_f$ is constructed using the $\breve{g}$-harmonic representative ψ of $[\psi]$ then $(\tilde{J}, \tilde{\Omega})$ defines an $SU(n)$-structure on (S, ∇^S).*

Proof. Since (M, J, g) is Calabi-Yau we have $h^{p,0}(X) = 0$ for $0 < p < \dim_{\mathbb{C}}(M, J)$ and $h^{0,0}(X) = h^{n,0}(X) = 1$. In particular, there is a Kähler class $[\breve{\omega}] \in H^2(M, \mathbb{Z})$ [Bea83, Prop 3.1]. Since (M, J) has trivial canonical bundle there is a Ricci-flat Kähler metric $\breve{g}$ with Kähler class $[\breve{\omega}] \in H^2(M, \mathbb{Z})$ by the Calabi-Yau theorem. By $\pi_1(M) = 0$ and [Bea83, Thm. 2.1] $(M, J, \breve{g})$ is isomorphic to a product of simply-connected Calabi-Yau manifolds and simple holomorphic symplectic manifolds. Using the values of the Hodge numbers $h^{p,q}$ we derive a contradiction unless $Hol(M, \breve{g}) = SU(\dim_{\mathbb{C}}(M, J))$. The remaining statements follow from Lemma 2.32 and Prop. 2.30 since $H^2(X, \mathbb{C}) = H^{1,1}(X)$. □

A $K3$-*surface* (M, J) is a simply connected Calabi-Yau manifold of complex dimension two. The Picard numbers of $K3$-surfaces range from 0 to 20. Hence, $(\tilde{J}, \tilde{\Omega})$ never defines an $SU(2)$-structure on a non-trivial toric type manifold over a projective $K3$-surface with $\rho(M, J) = 1$ since $H^{1,1}((M, J), \mathbb{Q})$ is generated by the Kähler class, i.e., $H^{1,1}_{prim}((M, J), \mathbb{Z}) = 0$. On the other hand, a $K3$-surface with maximal Picard number is called exceptional (cf. [ea04, Ch. VIII.8]) and we have

Corollary 2.34. *For an exceptional $K3$-surface (M, J) with Ricci-flat Kähler metric g and Kähler class $[\omega] \in H^2(M, \mathbb{Z})$ we have $rk(H^{1,1}_{prim}((M, J, g), \mathbb{Z})) = 19$ and if $(X, \tilde{g}_f)$ is of toric type over (M, J, g) where $c_1(\tilde{X} \to M) = -[\frac{\psi}{2\pi}] \in H^{1,1}_{prim}((M, J, g), \mathbb{Z})$ and $\tilde{g}_f$ is constructed using the harmonic representative ψ of $[\psi]$ then $(\tilde{J}, \tilde{\Omega})$ defines an $SU(2)$-structure on (S, ∇^S).*

Proof. Since (M, J) is a $K3$-surface we have $h^{2,0}(M, J) = h^{0,2}(M, J) = 1$ as well as $b_2(M, J) = 22$. Thus, Lemma 2.32 implies $rk(H^{1,1}_{prim}((M, J, g), \mathbb{Z})) = 19$ and the statement follows from Prop. 2.30. □

While being redundant, the preceding Corollary provides the idea for higher dimensional simple holomorphic symplectic manifolds. More precisely, on a simple holomorphic symplectic manifold $(M, \hat{J}, \hat{\sigma})$ Oguiso's theorem 1.23 implies the existence of a simple holomorphic symplectic structure (J, σ) on M such that $\rho(M, J) = b_2(M) - 2$. Examples of simple holomorphic symplectic manifolds such that $b_2(M) \geq 4$ can be found in [Bea83, Prop. 6].

Proposition 2.35. *Let (M, J) be a simple holomorphic symplectic manifold of complex dimension $2n$ such that $\rho(M, J) = b_2(M) - 2$ and $b_2(M) \geq 4$. Then there exists a hyperkähler structure $(J_1 = J, J_2, J_3, g)$ with Kähler class $[\omega] \in H^2(M, \mathbb{Z})$ on M and*

$0 \neq [\frac{\psi}{2\pi}] \in H^{1,1}(M,J) \cap H^{1,1}(M,J_2) \cap H^2(M,\mathbb{Z})$. Moreover, if $(X,\tilde{g}_f)$ is of toric type over (M,J,g) where $c_1(\tilde{X} \to M) = -[\frac{\psi}{2\pi}]$ and $\tilde{g}_f$ is constructed using the harmonic representative ψ of $[\psi]$ with $f \in C^\infty(X)$ suitable then

$$Hol(X,\tilde{g}_f) = \begin{cases} Sp(n) \ltimes \mathbb{R}^{4n} & \text{if } \frac{\partial f}{\partial x} \equiv 0, \\ (\mathbb{R}^* \times Sp(n)) \ltimes \mathbb{R}^{4n} & \text{otherwise.} \end{cases}$$

Proof. The first part is similar to [Joy00, Prop. 6.2.7]. Consider the $\mathbb{Q}$-vector subspace $W \subset H^2(M,\mathbb{Q})$ generated by $Im(Pic(M,J) \to H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{Q}))$. The image of W in $H^2(M,\mathbb{C})$ is contained in $H^{1,1}(M,J)$ by the Lefschetz theorem on $(1,1)$-classes and since $\dim_{\mathbb{Q}} W = \rho(M,J) = h^{1,1}(M,J)$ the image of W in $H^2(M,\mathbb{R})$ is dense in $H^{1,1}(M,J) \cap H^2(M,\mathbb{R})$. However, the set of Kähler classes $\mathcal{K}_{(M,J)}$ on (M,J) is a non-empty open cone in $H^{1,1}(M,J) \cap H^2(M,\mathbb{R})$ (cf. [Huy05, Ex. 3.25]). Therefore, $0 \neq \mathcal{K}_{(M,J)} \cap H^2(M,\mathbb{Q})$ and we derive a Kähler class $[\omega] \in H^2(M,\mathbb{Z})$ on (M,J). Moreover, using Beauville's theorem 1.22 we derive a hyperkähler structure $(J_1 = J, J_2, J_3, g)$ on M such that $[g(J(\cdot),(\cdot))] = [\omega]$ and $Hol(M,g) = Sp(n)$. If $[\frac{\psi}{2\pi}] \in H^{1,1}(M,J) \cap H^{1,1}(M,J_2) \cap H^2(M,\mathbb{Z})$ then $\psi \in \mathcal{A}^{1,1}(M,J_1) \cap \mathcal{A}^{1,1}(M,J_2)$ for the g-harmonic representative of $[\psi]$ and the statement follows from Prop. 2.23 and Prop. 2.25.

The existence of a non-trivial class in $H^{1,1}(M,J) \cap H^{1,1}(M,J_2) \cap H^2(M,\mathbb{Z})$ can be shown as follows. Define the operators

$$\mathrm{ad}J_i : \Lambda^{p,q}_{J_i} M \to \Lambda^{p+q} M \qquad \text{such that} \qquad \eta \mapsto (p-q)\sqrt{-1}\eta.$$

It is proved in [Ver95, Prop. 2.1] that the Lie algebra $\mathfrak{g}_M$ generated by $\mathrm{ad}J_1, \mathrm{ad}J_2, \mathrm{ad}J_3$ is isomorphic to $\mathfrak{su}(2)$ and that its action commutes with the Laplace operator. Using the Hodge-Theorem we derive an induced $\mathfrak{su}(2)$-action on the cohomology of M and if we let

$$H_{inv} := \{\alpha \in H^2(M,\mathbb{C}) : \mathfrak{g}_M \cdot \alpha = 0\}$$

then the definition implies $H_{inv} = H^{1,1}(M,J) \cap H^{1,1}(M,J_2)$ (cf. [Ver95, Prop. 2.2]). As explained in the proof of Prop. 5.2 in [Ver95] we have $H_{inv} \subset H^{1,1}_{prim}(M,J,g)$ by [Ver95, Claim 2.1] and $H^2(M,\mathbb{C})/H_{inv}$ is 3-dimensional, i.e., $H_{inv} = H^{1,1}_{prim}(M,J,g)$.

Since $\rho(M,J)$ is maximal Lemma 2.32 implies $rk(H^{1,1}_{prim}((M,J,g),\mathbb{Z}))) = b_2(M) - 3$ and we have a elements $c_1,\ldots,c_{b_2(M)-3} \in H^2(M,\mathbb{Z})$ inducing a basis of $H^{1,1}_{prim}(M,J,g) = H^{1,1}(M,J) \cap H^{1,1}(M,J_2)$. □

Let (M,g) be a compact quaternionic Kähler manifold with positive Ricci curvature. The results of LeBrun and Salamon (cf. Thm. 1.26) imply that $H^2(M,\mathbb{Z}) = 0$ unless (M,g) is isomorphic to the complex Grassmannian $Gr_2(\mathbb{C}^{n+2})$. Therefore, there is no non-trivial S^1-bundle over $M \neq Gr_2(\mathbb{C}^{n+2})$ and we refer to Example 2.26. On the other hand, if (M,g) is isomorphic to the complex Grassmannian $Gr_2(\mathbb{C}^{n+2})$ then $Hol(M,g) \neq Sp(m)Sp(1)$. The best we can do for negative quaternionic Kähler manifolds is summarized in

Corollary 2.36. *Let (M,g) be a quaternionic Kähler manifold such that $Hol(M,g) = Sp(n)Sp(1)$ and let $\pi : \mathcal{Z}_M \to M$ be its twistor space. Suppose $(X = \tilde{X} \times L, \tilde{g}_f)$ is of toric type over (M,g) where $c_1(\tilde{X} \to M) = -[\frac{\psi}{2\pi}] \in H^2(M,\mathbb{Z})$ and $\tilde{g}_f$ is constructed using a representative $\psi \in [\psi]$ with $f \in C^\infty(X)$ suitable. If $\pi^*(\psi) \in \mathcal{A}^{1,1}(\mathcal{Z}_M)$ then*

$$Hol(X,\tilde{g}_f) = \begin{cases} Sp(n)Sp(1) \ltimes \mathbb{R}^{4n} & \text{if } \frac{\partial f}{\partial x} \equiv 0, \\ (\mathbb{R}^* \times Sp(n)Sp(1)) \ltimes \mathbb{R}^{4n} & \text{otherwise.} \end{cases}$$

Proof. Fix $p \in \mathcal{Z}_M$. Let us briefly review the definition ([Bes87, 14.71]) of the integrable almost complex structure J on $\mathcal{Z}_M$. If $T_p\mathcal{Z}_M = \mathcal{V}_p \oplus \mathcal{H}_p$ is the decomposition of $T_p\mathcal{Z}_M$ into the vertical space $\mathcal{V}_p$ and the natural horizontal space $\mathcal{H}_p$ then $J\mathcal{V}_p = \mathcal{V}_p$ and $J\mathcal{H}_p = \mathcal{H}_p$. By the definition of the twistor space any $p \in \mathcal{Z}_M$ corresponds to a complex structure $\hat{J} \in E|_{\pi(p)}$ on $T_{\pi(p)}M$. Since π_* identifies $\mathcal{H}_p$ with $T_{\pi(p)}M$ we can define $J|_{\mathcal{H}_p} = \hat{J}$. Therefore, $\pi^*(\psi) \in \mathcal{A}^{1,1}(\mathcal{Z}_M, J)$ if and only if $\psi|_{\pi(p)}$ is a $(1,1)$-form w.r.t. any $\check{J} \in E|_{\pi(p)}$.

On the other hand, the real rank 3 bundle $E \subset End(TM)$ (cf. Prop. 1.25) induces a subbundle $\tilde{E}$ of $End(S)$ and we need to show $\nabla^S_Z \tilde{E} \subset \tilde{E}$.

However, $\nabla^S_Z J_i = 0$ for any locally defined almost complex structure $J_i \in \Gamma(U,E)$ is sufficient for $\nabla^S_Z \tilde{E} \subset \tilde{E}$ and the computations in the proof of Prop. 2.25ii imply $\psi|_{\pi(p)} \in T^{1,1}_{\pi(p)}(M, J_i|_{\pi(p)})$ for any $J_i \in \Gamma(U,E)$ if and only if $\nabla^S_Z J_i = 0$. □

Next, we show how to apply the preceding ideas to construct toric type manifolds with disconnected screen holonomy. Let X be a compact complex surface such that $b_1(X) = 0$ and write $K_X := \Omega^2_X$ for its canonical bundle. Then X is an *Enriques surface* if and only if $K_X \otimes K_X = \mathcal{O}_X$ and $K_X \neq \mathcal{O}_X$.

Any Enriques surface is projective and admits a 2-fold covering by a $K3$-surface. For more details on Enriques surfaces we refer to [ea04, Ch. VIII]. Since X is projective we can find a Kähler class $[\omega] \in H^2(M,\mathbb{Z})$. By the Calabi-Yau theorem there is a Ricci-flat Kähler metric g on X whose Kähler class is $[\omega]$.

Proposition 2.37. *Suppose that (M,J) is an Enriques surface with Ricci-flat Kähler metric g and Kähler form ω such that $[\omega] \in H^2(M,\mathbb{Z})$. Then there exists $0 \neq [\frac{\psi}{2\pi}] \in H^{1,1}_{prim}((M,J,[\omega]),\mathbb{Z})$ and if $(X = \tilde{X} \times L, \tilde{g}_f)$ is of toric type over (M,J,g) where $c_1(\tilde{X} \to M) = -[\frac{\psi}{2\pi}]$ and $\tilde{g}_f$ is constructed using the harmonic representative ψ of $[\psi]$ with $f \in C^\infty(X)$ suitable then there is a disconnected subgroup $G \subset U(2)$ whose identity component is $SU(2)$ such that*

$$Hol(X,\tilde{g}_f) = \begin{cases} G \ltimes \mathbb{R}^4 & \text{if } \frac{\partial f}{\partial x} \equiv 0, \\ (\mathbb{R}^* \times G) \ltimes \mathbb{R}^4 & \text{otherwise.} \end{cases}$$

Proof. Suppose we have shown $\mathfrak{hol}(\nabla^S) = \mathfrak{su}(2)$. Since $Hol(M,g) = Sp(1) \rtimes_{\mathbb{Z}_2} \mathbb{Z}_4 \subset U(2)$ (cf. [Bes87, 14.22]) for an Enriques surface with Ricci-flat Kähler metric Prop. 2.23 implies that $G = Hol(\nabla^S)$ has at least two components. Moreover, Prop. 2.25.ii implies $G \subset U(2)$ since $\psi \in \mathcal{A}^{1,1}(M,J)$.

For an Enriques surface (M, J) we have $H^2(M, \mathbb{C}) = H^{1,1}(M, J)$ and $Pic(M, J) \to H^2(M, \mathbb{Z}) = \mathbb{Z}^{10} \oplus \mathbb{Z}_2$ is an isomorphism (cf. [ea04, Ch. VIII.15]). In particular, $\rho(M, J) = 10$ is maximal, i.e., $[\omega] \in H^2(M, \mathbb{Z})$ and Lemma 2.32 imply

$$rk(H^{1,1}((M, J, g), \mathbb{Z})) = 9.$$

Hence, we may assume $0 \neq [\frac{\psi}{2\pi}] \in H^{1,1}_{prim}((M, J, [\omega]), \mathbb{Z})$.

It remains to show $\mathfrak{hol}(\nabla^S) = \mathfrak{su}(2)$. Let $F : (N, F^*g) \to (M, g)$ be the universal covering map. Then (N, F^*g) is a $K3$-surface and F is holomorphic. Consider the pullback $F^*\pi : \hat{X} \to N$ of the bundle $\pi : \tilde{X} \to M$ along F and the diagram

$$\begin{array}{ccc} (\hat{X} \times L, (\tilde{F} \times id_L)^* \tilde{g}_f) & \stackrel{\tilde{F} \times id_L}{\longrightarrow} & (\tilde{X} \times L, \tilde{g}_f) \\ {\scriptstyle F^*\pi} \downarrow & & \downarrow {\scriptstyle \pi} \\ (N, F^*g) & \stackrel{F}{\longrightarrow} & (M, g) \end{array}$$

Thus, $\hat{X} \times L$ covers $\tilde{X} \times L$. Moreover, $(\hat{X} \times L, (\tilde{F} \times id_L)^* \tilde{g}_f)$ coincides with the toric type manifold over (N, F^*g) which is constructed using $c_1(\hat{X} \to N) = -[\frac{F^*\psi}{2\pi}] \in H^2(M, \mathbb{Z})$ and the representative $F^*\psi$ of $[F^*\psi]$. Hence, the screen holonomy of $\tilde{F}^* \tilde{g}_f$ equals $SU(2)$ if $F^*\psi$ is F^*g-harmonic and $[\frac{F^*\psi}{2\pi}] \in H^{1,1}_{prim}((N, F^*g), \mathbb{Z})$.

However, $F^*\psi$ is F^*g-harmonic since ψ is g-harmonic and F is a Riemannian covering map. Since F is holomorphic the pullback of a $(1,1)$-form is of type $(1,1)$, i.e., $[\frac{F^*\psi}{2\pi}] \in H^{1,1}(N, \mathbb{Z})$. By definition the Kähler class of (N, F^*g) is given by $\hat{\omega} := F^*[\omega]$ and on any complex Kähler surface $(Y, [\alpha])$ we have $[\beta] \in H^2_{prim}(Y, [\alpha])$ if and only if $[\alpha \wedge \beta] = 0$. Finally, $\hat{\omega} \wedge [F^*\psi] = F^*([\omega \wedge \psi]) = 0$ since $[\psi] \in H^{1,1}_{prim}(M, J, g)$. □

Remark 2.38. An upper bound for the number of components of G in Prop. 2.37 is given by the fundamental group $\pi_1(\tilde{X})$ which (using Remark 2.19) is abelian and appears in the exact sequence

$$\underbrace{\mathbb{Z}^{10} \oplus \mathbb{Z}_2}_{=H^2(M,\mathbb{Z})} \stackrel{-[\frac{\psi}{2\pi}]\cup\cdot}{\longrightarrow} \underbrace{\mathbb{Z}}_{=H^4(M,\mathbb{Z})} \stackrel{\pi^*}{\longrightarrow} \underbrace{\pi_1(\tilde{X})}_{=H^4(\tilde{X},\mathbb{Z})} \stackrel{\pi_*}{\longrightarrow} \underbrace{\mathbb{Z}_2}_{=H^3(M,\mathbb{Z})} \to 0.$$

Hence, $Hol(\nabla^S)$ has finitely many components and since we have shown $Hol(\nabla^S) \subset U(2)$ the group G must be of the form $\mathbb{Z}_r \cdot SU(2)$, r odd or $\mathbb{Z}_{2r} \cdot SU(2)$, r even (cf. [McI91, Sec. V.]). □

Definition 2.39. *A Lorentzian manifold (X, g) is a pp-wave if $\mathfrak{hol}(X, g) = \mathbb{R}^{\dim X - 2}$, i.e., if $\mathfrak{hol}(X, g)$ acts weakly irreducibly and ∇^S is flat.*

Our definition of pp-waves coincides with [Lei06]. Note, however, that there are non-equivalent definitions of pp-waves in the literature.

Proposition 2.40. *For $n \geq 2$ let $T^n = S^1 \times \ldots \times S^1$ be the n-dimensional torus and let g be the flat Riemannian metric on T^n such that $y^i \mapsto e^{2\pi\sqrt{-1}y^i}$ induce the global* *orthonormal frame* $(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n})$ *on* TT^n. *If* $(X = \tilde{X} \times L, \tilde{g}_f)$ *is of toric type over* (T^n, g) *where* $c_1(\tilde{X} \to M) = -[dy^1 \wedge dy^2]$ *is induced by the volume form of* $T^2 \hookrightarrow T^n$ *and* $\tilde{g}_f$ *is constructed using* $\psi := 2\pi dy^1 \wedge dy^2$ *with* $f \in C^\infty(X)$ *suitable then*

$$\mathfrak{hol}(X, \tilde{g}_f) = \begin{cases} \mathbb{R}^n & \text{if } \frac{\partial f}{\partial x} \equiv 0, \\ \mathbb{R} \ltimes \mathbb{R}^n & \text{otherwise.} \end{cases}$$

Proof. This follows from the first part of Prop. 2.25 since (T^n, g) is flat and $\nabla^{(T^n,g)}\psi = 0$. □

The possible weakly irreducible holonomy algebras in dimension 3 are $\mathbb{R}$, $\mathbb{R} \ltimes \mathbb{R}$ and $\mathfrak{so}(1,2)$. We conclude

Corollary 2.41. *Let* (T^2, g) *be the flat torus with standard local coordinates* (y^1, y^2). *Let* $\eta := dy^2$ *and define* $X \to T^2$ *using the volume form. If* $\tilde{g}_f$ *is the Lorentzian metric from Prop. 2.17 with* $f \in C^\infty(X)$ *suitable then*

$$\mathfrak{hol}(X, \tilde{g}_f) = \begin{cases} \mathbb{R} & \text{if } \frac{\partial f}{\partial x} \equiv 0, \\ \mathbb{R} \ltimes \mathbb{R} & \text{otherwise.} \end{cases}$$

□

Proposition 2.42. *Let* T^{n+1} *be the flat torus with* $\psi := 2\pi dy^1 \wedge dz$ *and* $\eta := dz$ *where the coordinates* $(y^1, \ldots, y^n, z)$ *on* T^{n+1} *are given by* $y^i \mapsto e^{2\pi\sqrt{-1}y^i}$. *If* $(X, \tilde{g}_f)$ *is constructed as in Prop. 2.17 with* $f \in C^\infty(T^{n+1})$ *suitable then* $(X, \tilde{g}_f)$ *is a complete compact pp-wave.*

Proof. We have to show that the $\tilde{g}_f$-geodesics are defined for all $t \in \mathbb{R}$. Our approach is motivated by [CFS03]. Let $F_{n+1} : \mathbb{R}^{n+1} \to T^{n+1}$ be the universal covering map and consider the diagram

$$\begin{array}{ccc} (\mathbb{R} \times \mathbb{R}^{n+1}, (\tilde{F}_{n+1} \circ F_1 \times \mathrm{id})^* \tilde{g}_f) & & \\ {\scriptstyle F_1 \times \mathrm{id}} \downarrow & & \\ (S^1 \times \mathbb{R}^{n+1}, \tilde{F}_{n+1}^* \tilde{g}_f) & \xrightarrow{\tilde{F}_{n+1}} & (X, \tilde{g}_f) \\ {\scriptstyle pr_{\mathbb{R}^{n+1}}} \downarrow & & \downarrow {\scriptstyle \pi} \\ \mathbb{R}^{n+1} & \xrightarrow{F_{n+1}} & T^{n+1} \end{array}$$

We write $\check{g} := (\tilde{F}_{n+1} \circ F_1 \times \mathrm{id})^* \tilde{g}_f$. Then

$$\check{g} = 2dxdz + (4\pi y^1 + f + 1)dz^2 + \sum_{i=1}^n (dy^i)^2.$$

Let $\gamma(t) = (x(t), y^i(t), z(t))$ be a curve on $\mathbb{R}^{n+2}$ of constant energy $E_\gamma := g(\dot{\gamma}, \dot{\gamma})$. We compute

$$0 = \ddot{z} + \Gamma_{ij}^{n+1}\dot{y}^i\dot{y}^j = \ddot{z},$$
$$0 = \ddot{y}^i + \Gamma_{jk}^{i}\dot{y}^j\dot{y}^k = \ddot{y}^i - \frac{\dot{z}^2}{2}(\frac{\partial f}{\partial y^i} + 4\pi\delta_1^i).$$

Hence $\dot{z} =: A$ is constant. Let γ_2 be the projection of γ to $\mathbb{R}^n \subset \mathbb{R}^{n+2}$ given by the (y^i) coordinates. Then

$$\frac{\nabla_{\langle\cdot,\cdot\rangle}}{dt}\dot{\gamma}_2 = \frac{A^2}{2}(\mathrm{grad}_{\langle\cdot,\cdot\rangle} f + 4\pi\frac{\partial}{\partial y^1}) \qquad \text{on } (\mathbb{R}^n, \langle\cdot,\cdot\rangle). \tag{2.1}$$

Suppose γ_2 is defined for all $t \in \mathbb{R}$. Since $E_\gamma = 2\dot{x}\dot{z} + (4\pi y^1 + f + 1)\dot{z}^2 + \sum_{i=1}(\dot{y}^i)^2$ we conclude $x(t) = \dot{x}(0)t + x_0$ if $A = 0$ and

$$x(t) = x_0 + \frac{1}{2A}\int_0^t E_\gamma - g(\dot{\gamma}_2, \dot{\gamma}_2) - A^2(f(\gamma_2(s)) + 1 + 4\pi y^1(s))ds$$

otherwise. In order to show the existence of γ_2 for all $t \in \mathbb{R}$ we define $\alpha(t) := (\gamma_2, \dot{\gamma}_2)$ and

$$F(x_1, \ldots, x_{2n}) := (x_n, \ldots, x_{2n}, \frac{A^2}{2}(\partial_1 f + 4\pi), \frac{A^2}{2}\partial_2 f, \ldots, \frac{A^2}{2}\partial_n f).$$

Then equation (2.1) is equivalent to $\dot{\alpha} = F(\alpha)$.

Define $C := \sup_{T^{n+1}}|f| + \sup_{T^{n+1}}|\nabla f|$. If α is not defined for all $t \in \mathbb{R}$ then it must leave any compact set. However, $\alpha(t) = \alpha(t_0) + \int_{t_0}^t F(\alpha(s))ds$ and Gronwall's lemma imply that α is bounded on any $[t_0, t_1)$ since

$$\|F(x)\|^2 \le \sum_{j=1}^n (x_{j+n})^2 + \frac{A^4}{4}(C^2 + 2C + 1) \le \|x\|^2 + \frac{A^4}{4}(C^2 + 2C + 1).$$

Hence, γ_2 is defined for all $t \in \mathbb{R}$. □

As we will see in the next section any Lorentzian manifold admitting a parallel lightlike vector field is covered by a product $\mathbb{R} \times M$ once we impose a certain completeness condition. Therefore, we finish this section with two examples.

Example 2.43. *The Lorentzian manifold $(X, \tilde{g}_f)$ from Cor. 2.41 is not diffeomorphic to a product.*

Proof. If $X = S^1 \times Y$ then $b_1(X) = 1 + b_1(Y)$ and $b_2(X) = b_2(Y) + b_1(Y)$. However, Gysin's sequence implies $b_1(X) = b_2(X) = 2$ since $0 \neq c_1(X \to T^2) \in H^2(T^2, \mathbb{R})$, i.e., $b_2(Y) = b_1(Y) = \chi(Y) = 1$. Thus, the classification of closed surfaces implies a contradiction. □

Example 2.44. *Let $M := \mathbb{R}^3 \setminus \{(0,0,-1), (0,0,+1)\}$ and let $[\frac{\psi}{2\pi}] \in H^2(M, \mathbb{Z})$ be a generator. Define $\eta := \frac{\partial}{\partial z}$ on M and construct $(X, \tilde{g}_f)$ as in Prop. 2.17 with $f \in C^\infty(X)$*

suitable. Then $\pi_1(X)$ *is finite and* X *is not diffeomorphic to* $S^1 \times Y$ *or* $\mathbb{R} \times Y$*. Moreover,* $\mathfrak{hol}(X, \tilde{g}_f)$ *is weakly irreducible with index 1.*

Proof. M is homotopic to a wedge sum of two 2-spheres, i.e., $\pi_1(M) = 0$, $\pi_2(M) = \mathbb{Z}^2$ and $H^3(M, \mathbb{R}) = 0$. By Gysin's sequence $H^1(M) \to H^1(X) \to H^0(M) \to H^2(M)$, i.e., $b_1(X) = 0$. The homotopy sequence of the fibration implies $\pi_2(M) \to \pi_1(S^1) \to \pi_1(X) \to \pi_1(M) = 0$, i.e., $\pi_1(X) = H_1(X, \mathbb{Z})$ is abelian. Hence, $\pi_1(X)$ is finite and $X \neq S^1 \times Y$. Using Gysin's sequence once again we have $0 = H^3(M) \to H^3(X) \to H^2(M) = \mathbb{R}^2 \to H^4(M) = 0$, i.e., $b_3(X) = 2$. Thus, $X = \mathbb{R} \times Y$ implies the contradiction $2 = b_3(Y) \in \{0, 1\}$. □

2.3 Decent and Horizontal Spacetimes

In this section we apply foliation theory to study the geometry of weakly irreducible Lorentzian manifolds. In order to avoid confusion we write g^L and ∇^L for the Lorentzian metric and its Levi-Civita connection.

All Lorentzian manifolds studied in this and the next section are supposed to be orientable manifolds!

Definition 2.45. *Let (X, g^L) be a Lorentzian manifold and $V \in \Gamma(X, TX)$ a global nowhere vanishing lightlike vector field. We say (X, g^L, V) is an*

1. *almost decent spacetime if $\nabla^L_\cdot V = \alpha(\cdot)V$ for some 1-form $\alpha \in \Gamma(X, T^*X)$.*
2. *decent spacetime if it is almost decent and $\alpha|_{\Xi^\perp} = 0$, where $\Xi := span\{V\}$.*

If (X, g^L) admits a parallel lightlike subbundle $\Xi \subset X$ of rank one then we have a foliation $\mathcal{X}^\perp$ of codimension one which is induced by $\Xi^\perp \supset \Xi$. Moreover, Ξ induces a foliation $\mathcal{X}$ of dimension one on X and if $\mathcal{L}^\perp$ is a leaf of $\mathcal{X}^\perp$ then $\Xi|_{\mathcal{L}^\perp}$ induces a 1-dimensional foliation $\mathcal{X}|_{\mathcal{L}^\perp}$ on $\mathcal{L}^\perp$.

Next, we review which Lorentzian manifolds are almost decent. For an arbitrary Lorentzian manifold (X, g^L) let $\mathfrak{hol}_p(X, g^L)$ be its holonomy algebra at $p \in X$. Suppose the Borel-Lichnérowicz decomposition is given by

$$T_pX = E_0 \oplus \ldots \oplus E_\ell \quad \text{and} \quad \mathfrak{hol}_p(X, g^L) = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_\ell,$$

i.e., all E_i are non-degenerate subspaces and each $\mathfrak{h}_j$ acts weakly irreducibly on E_j and trivially on E_i for $i \neq j$. W.l.o.g. we may assume that either E_0 or E_1 is not positive definite. Using Thm. 1.9 we derive three possible cases:

1. $E_0 = 0$ or $g^L|_{E_0}$ is positive definite and $\mathfrak{h}_i$ acts irreducibly for $i \geq 1$. Then $\mathfrak{h}_1 = \mathfrak{so}(1, n+1)$ for $n \geq 1$ and $\mathfrak{h}_j$ acts as an irreducible Riemannian holonomy representation for $j \geq 2$.
2. $E_0 \neq 0$ and $g^L|_{E_0}$ is negative definite or of Lorentzian signature. Thus, $\mathfrak{h}_j$ acts as an irreducible Riemannian holonomy representation for $j \geq 1$.
3. $E_0 = 0$ or $g^L|_{E_0}$ is positive definite and $\mathfrak{h}_1 \subset \mathfrak{so}(1, n+1)$ is weakly irreducible with index 1. Hence, $\mathfrak{h}_j$ acts as an irreducible Riemannian holonomy representation for $j \geq 2$.

In the first case, $\mathfrak{hol}(X, g^L)$ does not leave any lightlike line invariant, i.e., (X, g^L) cannot be almost decent. There is no general statement for the second case. For the third case Lemma 2.10 implies that $Hol(X, g^L)$ leaves a lightlike line $\mathbb{R} \cdot v$ invariant if $\mathfrak{h}_1 \neq \mathfrak{so}(1, 1)$. Let $\Xi \subset TX$ be the vector bundle corresponding to $\mathbb{R} \cdot v$. By Lemma 2.14 Ξ admits a global nowhere vanishing section V if and only if (X, g^L) is time-orientable. By the holonomy principle $\nabla^L_\cdot V = \alpha(\cdot)V$ for some 1-form $\alpha \in \Gamma(X, T^*X)$.

As we have already explained in Remark 2.8 the screen bundle of a Lorentzian manifold (X, g^L) admitting a lightlike parallel subbundle $\Xi \subset TX$ of rank one is defined as $\mathcal{S} := \mathrm{Coker}(\Xi \hookrightarrow \Xi^\perp)$ and each non-canonical splitting $s : \mathcal{S} \to \Xi^\perp$ of the exact sequence

$$0 \longrightarrow \Xi \longrightarrow \Xi^\perp \longrightarrow \mathcal{S} \longrightarrow 0$$

defines a realization $S := s(\mathcal{S}) \subset TX$. Moreover, using Cor. 2.6 a choice of a realization corresponds to a uniquely defined lightlike subbundle $\Theta \subset S^\perp$ of rank one with the following property: If $V \in \Gamma(U \subset X, \Xi)$ is nowhere vanishing then there exists a unique section $Z \in \Gamma(U, \Theta)$ such that $g^L(V, Z) = 1$.

Suppose the Lorentzian manifold (X, g^L) has a holonomy representation as in the third case with Ξ as above. We conclude that the following are equivalent.

- Ξ admits a global section V such that (X, g^L, V) is almost decent,
- (X, g^L) is time-orientable,
- $\mathcal{X}^\perp$ is transversely orientable.

Remark 2.46. If (X, g^L, V) is almost decent we always assume that $V \in \Gamma(X, \Xi)$ is future pointing. $\square$

Let (X, g^L, V) be an almost decent spacetime and let S be a realization of the screen bundle. If $Z \in \Gamma(X, \Theta)$ is the uniquely defined vector field from above then the (V, S)-metric associated to g^L is defined to be the following Riemannian metric on X.

$$g^R(A, B) := \begin{cases} 1 & \text{if } A = B = V \text{ or } A = B = Z, \\ g^L(A, B) & \text{if } A, B \in S, \\ 0 & \text{if } A \in S,\ B \in \{V, Z\} \text{ or } A = V,\ B = Z. \end{cases}$$

The following two observations are key results for the whole section.

Lemma 2.47. *Let (X, g^L, V) be an almost decent spacetime. For any realization S of the screen bundle the following are equivalent.*

1. *The (V, S)-metric g^R is bundle-like w.r.t. $\mathcal{X}^\perp$ and $(X, \mathcal{X}^\perp)$ is transversely parallelizable, i.e., there is $Z \in \Gamma(X, TX)$ such that $[T\mathcal{X}^\perp, Z] \subset T\mathcal{X}^\perp$ and $TX/_{T\mathcal{X}^\perp} = span\{pr_{TX/T\mathcal{X}^\perp}(Z)\}$,*
2. *the 1-form $g^L(V, \cdot)$ defining $\mathcal{X}^\perp$ is closed and*
3. *(X, g^L) is decent, i.e., $\alpha|_{\Xi^\perp} = 0$.*

Proof. Suppose $\alpha|_{\Xi^\perp} = 0$. Let $V \in \Gamma(X, \Xi)$ and fix $Z \in \Gamma(X, \Theta)$. We have to show $(L_W g^R)(Z, Z) = 0$ for all $W \in \Gamma(U, \Xi^\perp)$. Using $g^R(\cdot, Z) = g^L(\cdot, V)$ we derive

$g^R(\nabla^L_W Z, Z) = g^L(\nabla^L_W Z, V) = -g^L(Z, \nabla^L_W V) = -\alpha(W)$ and

$$\begin{aligned}(L_W g^R)(Z,Z) &= \underbrace{W(g^R(Z,Z))}_{=0} - 2g^R([W,Z],Z)\\ &= 2g^R(\underbrace{\nabla^L_Z W}_{\in \Xi^\perp}, Z) - 2g^R(\nabla^L_W Z, Z)\\ &= 2\alpha(W).\end{aligned}$$

Thus, g^R is bundle-like w.r.t. $\mathcal{X}^\perp$. Moreover, $pr_Z([W,Z]) := g^R([W,Z],Z)Z = -\alpha(W)Z$ and Z is globally defined, i.e., the foliation $\mathcal{X}^\perp$ is transversely parallelizable. Next, we compute

$$\begin{aligned}d(g^L(V,\cdot))(W,Z) &= g^L(\nabla^L_W V, Z) - g^L(\nabla^L_Z V, W)\\ &= \alpha(W)g^L(V,Z) - \alpha(Z)g^L(V,W) = \alpha(W).\end{aligned}$$

For the converse we follow these equations backwards. □

Lemma 2.48. *Let (X, g^L, V) be an almost decent spacetime and let $\mathcal{L}^\perp$ be a leaf of $\mathcal{X}^\perp$. Fix any realization S of the screen bundle.*

1. *The restriction $g^R|_{\mathcal{L}^\perp}$ of the (V,S)-metric is bundle-like w.r.t. $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$.*

2. *The (V,S)-metric is bundle-like w.r.t. the foliation $(X,\mathcal{X})$ if and only if $\alpha(V) = 0$ and $[V,Z] \in \Gamma(X,\Xi)$.*[16]

Proof. Since $g^R|_{S\times S} = g^L|_{S\times S}$ we have for any $Y_1, Y_2 \in \Gamma(U,S)$

$$\begin{aligned}(L_V g^R)(Y_1,Y_2) &= V(g^L(Y_1,Y_2)) - g^L([V,Y_1],Y_2) - g^L([V,Y_2],Y_1)\\ &= g^L(\nabla^L_V Y_1, Y_2) + g^L(Y_1, \nabla^L_V Y_2)\\ &\quad - g^L([V,Y_1],Y_2) - g^L([V,Y_2],Y_1)\\ &= g^L(\nabla^L_{Y_1} V, Y_2) + g^L(Y_1, \nabla^L_{Y_2} V) = 0.\end{aligned}$$

For the second statement we need to show $(L_V g^R)(Y_1,Y_2) = 0$ for any $Y_1, Y_2 \in \Gamma(U, S \oplus \Theta)$. If $Y_1 = Z$ and $Y_2 \in \Gamma(U,S)$ we derive

$$\begin{aligned}(L_V g^R)(Z,Y_2) &= V(g^R(Z,Y_2)) - g^R([V,Z],Y_2) - g^R([V,Y_2],Z)\\ &= -g^L([V,Z],Y_2) - g^L(\underbrace{[V,Y_2]}_{\in\Xi^\perp},V) = -g^L([V,Z],Y_2).\end{aligned}$$

If $Y_1 = Y_2 = Z$ then $(L_V g^R)(Z,Z) = V(g^R(Z,Z)) - 2g^R([V,Z],Z) = 2\alpha(V)$. Since $g^L(\nabla^L_V Z, V) = -\alpha(V)$ we conclude $[V,Z] \in \Gamma(X,\Xi)$ if $\alpha(V) = 0$ and $[V,Z] \in \Gamma(X, \Xi \oplus \Theta)$. □

[16] Since $\nabla^L_\cdot V = \alpha(\cdot)V$ the integral curves of V are g^L-geodesics if and only if $\alpha(V) = 0$.

Remark 2.49. Lemma 2.47 and Lemma 2.48 seem to be known. The only reference I could find is [Zeg99], where the first part of Lemma 2.48 has been used by Zeghib to study 3-dimensional Lorentzian manifolds. □

The following statement is an immediate consequence of Conlon's results presented in Prop. 1.38.

Corollary 2.50. *For a decent spacetime (X, g^L, V) all leaves of $(X, \mathcal{X}^\perp)$ have trivial leaf holonomy. Suppose there is a realization of the screen bundle such that Z is complete*[17] *and let $\mathcal{L}^\perp$ be a leaf of $\mathcal{X}^\perp$.*

1. *If there is no leaf of $\mathcal{X}^\perp$ which is closed in X then each leaf is dense in X.*

2. *We have $\tilde{X} = \tilde{\mathcal{L}^\perp} \times \mathbb{R}$ where $\tilde{X}$, $\tilde{\mathcal{L}^\perp}$ denote the universal covers of X, $\mathcal{L}^\perp$.*

3. *If there is a closed leaf then $X \to X/_{\mathcal{X}^\perp}$ is a smooth fiber bundle and $X/_{\mathcal{X}^\perp} \in \{\mathbb{R}, S^1\}$.*

4. *The inclusion $\mathcal{L}^\perp \to X$ induces a monomorphism $\pi_1(\mathcal{L}^\perp) \to \pi_1(X)$ onto a normal subgroup. If X is compact then $\pi_1(X)/_{\pi_1(\mathcal{L}^\perp)} = \mathbb{Z}^r$ for some $r \geq 1$ and $r = 1$ if and only if $\mathcal{L}^\perp$ is closed in X.*

Proof. The only statement not appearing in Prop. 1.38 is that about the leaf holonomy. However, we have shown that $(X, \mathcal{X}^\perp)$ is transversely parallelizable and for such a foliation each leaf has trivial leaf holonomy [Mol88]. □

We say that a piecewise smooth curve is causal if each tangent vector is causal such that the two tangent vectors at breakpoints are elements of the same half of the causal cone.

Definition 2.51. *Let (X, g^L) be a time-orientable Lorentzian manifold and $p \in X$. We say (X, g^L) is*

1. *distinguishing at $p \in X$ if for any neighborhood $U \ni p$ there is a neighborhood $V \subset U$ containing p such that any piecewise smooth causal curve $\gamma : [a, b] \to X$ with $\gamma(a) = p$ and $\gamma(b) \in V$ is contained in V.*

2. *strongly causal at $p \in X$ if for any neighborhood $U \ni p$ there is a neighborhood $V \subset U$ containing p such that any piecewise smooth causal curve $\gamma : [a, b] \to X$ with $\gamma(a), \gamma(b) \in V$ is contained in U.*

3. *a causal spacetime if there is no piecewise smooth causal curve $\gamma : [a, b] \to X$ such that $\gamma(a) = \gamma(b)$.*

4. *a strongly causal resp. distinguishing spacetime if it is strongly causal resp. distinguishing for all $p \in X$.*

[17] The integral curves of Z are g^R-geodesics by the Koszul-formula, i.e., if g^R is complete so is Z.

5. *a stably causal spacetime if it admits a temporal function, i.e., $f \in C^\infty(X)$ such that $grad_{g^L}(f)$ is timelike and past-directed.*

At this point we remind of the causality ladder (cf. [MS08a]) of a spacetime (X, g^L) on which we have the implications

$$\text{stably causal} \Longrightarrow \text{strongly causal} \Longrightarrow \text{distinguishing} \Longrightarrow \text{causal.}$$

Proposition 2.52. *Let (X, g^L, V) be an almost decent spacetime.*

1. *If (X, g^L) is causal then the leaves of the foliated manifolds $(X, \mathcal{X})$ and $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ have trivial leaf holonomy. Moreover, X is not compact.*
2. *If (X, g^L) is distinguishing at $p \in X$ then the leaf $\mathcal{L}^\perp$ of $\mathcal{X}^\perp$ through p is not compact.*
3. *If (X, g^L) is strongly causal at $p \in X$ and $\mathcal{L}^\perp$ is the leaf $\mathcal{L}^\perp$ of $\mathcal{X}^\perp$ through p then $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ is regular at p.*
4. *If (X, g^L) is strongly causal then each leaf of $\mathcal{X}$ is a closed subset in X and each leaf of $\mathcal{X}|_{\mathcal{L}^\perp}$ is a closed subset in $\mathcal{L}^\perp$.*
5. *If (X, g^L) is stably causal then there exists an integrable realization of the screen bundle.*

Proof.

1. It is well known that a causal spacetime (X, g^L) is non-compact. Any curve in a leaf $\mathcal{L}$ of $\mathcal{X}$ is lightlike, i.e., $\pi_1(\mathcal{L}) = 0$ since (X, g^L) is causal.
2. Suppose $\mathcal{L}^\perp$ is compact. By Lemma 2.48 there is a bundle-like Riemannian metric on the compact foliated manifold $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$. Consider the leaf $\mathcal{L} \subset \mathcal{L}^\perp$ of $\mathcal{X}|_{\mathcal{L}^\perp}$ through p. If $\mathcal{L}$ is closed in $\mathcal{L}^\perp$ then it is compact, i.e., we have a closed lightlike curve through p. In this case, (X, g^L) would not be causal at p. On the other hand, if $\mathcal{L}$ is not closed in $\mathcal{L}^\perp$ then its closure $\bar{\mathcal{L}} \subset \mathcal{L}^\perp$ is diffeomorphic to a torus in $\mathcal{L}^\perp$ by Carrière's theorem 1.31. In particular, there is a lightlike curve γ tangent to $\mathcal{L}$ such that $\overline{Im(\gamma)}$ is a torus. Hence, (X, g^L) is not distinguishing at p.
3. Fix a Walker coordinate neighborhood $\tilde{U} \ni p$, i.e., $g^L = 2dxdz + 2u_\alpha dy^\alpha dz + g_{\alpha\beta}dy^\alpha dy^\beta$ in $\tilde{U}$ and $\Xi|_{\tilde{U}} = \text{span}\{\partial_x\}$ such that $x(p) = y^i(p) = z(p) = 0$. Since (X, g^L) is strongly causal at p there is $V \subset U$ as in Def. 2.51. Hence, there is $\varepsilon > 0$ such that $U := \{q \in \tilde{U} : |x(q)|, |y^i(q)|, |z(q)| < \varepsilon\} \subset V$ and $\hat{U} := \{q \in U : z(q) = z(p)\}$ is open in $\mathcal{L}^\perp$ and a flat coordinate neighborhood of $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$. Suppose there is a leaf $\mathcal{L}$ of $\mathcal{X}|_{\mathcal{L}^\perp}$ such that $q_a, q_b \in \hat{U} \cap \mathcal{L}$. Let $\gamma : [a, b] \to \mathcal{L}$ be an integral curve of V such that $\gamma(a) = q_a$ and $\gamma(b) = q_b$. Since (X, g^L) is strongly causal at p we have $\gamma([a, b]) \subset \tilde{U}$ and as γ is tangent to Ξ we have $\gamma([a, b]) \subset \{(\cdot, y^1(\gamma(a)), \dots, y^n(\gamma(b)), z(p))\}$. By definition of $\hat{U}$, q_a and q_b can be connected by a curve contained in $\hat{U} \cap \mathcal{L}$. Therefore, each non-empty intersection of $\hat{U}$ with a leaf of $\mathcal{X}|_{\mathcal{L}^\perp}$ is connected.

4. If (X, g^L) is strongly causal then $\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp}$ is a regular foliation and for such foliations it is well known that all leaves are closed. However, for completeness and in order to show that all leaves of $\mathcal{X}$ are closed in X here is a direct proof:

   Let $\mathcal{L} \subset \mathcal{L}^\perp$ be a leaf of $\mathcal{X}$. Suppose we have $q \in \bar{\mathcal{L}} \setminus \mathcal{L}$, where the closure is taken w.r.t. X. Fix a Walker coordinate neighborhood $\tilde{U}$, i.e., $g^L = 2dxdz + 2u_\alpha dy^\alpha dz + g_{\alpha\beta} dy^\alpha dy^\beta$ in $\tilde{U}$ and $\Xi|_{\tilde{U}} = \operatorname{span}\{\partial_x\}$ whereas ∂_x is future pointing. If $x(q) = y^i(q) = z(q) = 0$ then there is $\varepsilon > 0$ such that $\bar{U} \subset \tilde{U}$ for $U := \{p \in \tilde{U} : |x(p)|, |y^i(p)|, |z(p)| < \varepsilon\}$. Suppose there is $V \subset U$ as in Def. 2.51.

   Since $q \in \bar{\mathcal{L}} \setminus \mathcal{L}$ there is $p_V = (x_V, y_V^1, \dots, y_V^n, z_V) \in V \cap \mathcal{L}$. Let γ be a lightlike future directed curve generating $\mathcal{L}$ such that $\gamma(0) = p_V$. Fix $x_1 < x_V < x_{-1}$ such that $(x_{\pm 1}, y_V^1, \dots, y_V^n, z_V) \in \tilde{U} \setminus \bar{U}$. Since $(\cdot, y_V^1, \dots, y_V^n, z_V) \in \mathcal{L}$ there are $t_{-1} < 0 < t_1$ such that $\gamma(t_{\pm 1}) = (x_{\pm 1}, y_V^1, \dots, y_V^n, z_V)$. As above we can find

$$\tilde{p}_V \in \mathcal{L} \cap (V \setminus \{(\cdot, y_V^1, \dots, y_V^n, z_V)\}).$$

   Moreover, $\tilde{p}_V = \gamma(\tilde{t})$ for some $\tilde{t}$. If $\tilde{t} < 0$ then $\tilde{t} < t_{-1}$ and $\gamma|_{[\tilde{t},0]}$ is a future pointing lightlike curve leaving U with endpoints in V. Otherwise $\tilde{t} > 0$ and therefore $\tilde{t} > t_1$, i.e., $\gamma|_{[0,\tilde{t}]}$ is a future pointing lightlike curve leaving U with endpoints in V. In this case, (X, g^L) would not be strongly causal at q. Hence, $\mathcal{L}$ is closed in X and being the preimage of a closed set under $\mathcal{L}^\perp \to X$ it is closed in $\mathcal{L}^\perp$.

5. If $f \in C^\infty(X)$ has past directed timelike gradient $T \in \Gamma(X, TX)$ then $g^L(V,T) > 0$ and we define $S := \operatorname{span}\{V, T\}^{\perp_{g^L}}$, i.e., $Z = \frac{1}{g^L(V,T)} T - \frac{g^L(T,T)}{2g^L(V,T)^2} V$. Given $Y_1, Y_2 \in \Gamma(U, S)$ we have $[Y_1, Y_2] \in \Gamma(U, \Xi^\perp)$ and since $Y_1, Y_2 \in \Gamma(U, \ker df)$ we have $[Y_1, Y_2] \in \Gamma(U, \ker df)$. Therefore, $[Y_1, Y_2] \in \Gamma(U, \Xi^\perp \cap \ker df = S)$.

□

Example 2.53. *Let (M, g) be a simply connected compact Riemannian manifold and $f \in C^\infty(M)$. For $\varepsilon > 0$ and $L \in \{\mathbb{R}, S^1\}$ define $X := S^1 \times L \times M$ and*

$$g_\varepsilon^L := 2dxdz + \varepsilon f dz^2 + g,$$

where dx and dz are the coordinate 1-forms on $S^1 \times L$. If $f \in C^\infty(M)$ is suitable then (X, g_ε^L) is weakly irreducible whereas ∂_x is $\nabla^{g_\varepsilon^L}$-parallel. Moreover, the leaves of $(X, \mathcal{X}^\perp)$ are compact and the universal cover of (X, g_ε^L) is globally hyperbolic if ε is sufficiently small.

Proof. Each leaf of $\mathcal{X}^\perp$ is diffeomorphic to $S^1 \times M$ and the universal cover of X is given by $\mathbb{R}^2 \times M$. The pullback of g_ε^L to $\mathbb{R} \times M$ is of the form $2dxdz + \varepsilon f dz^2 + g$ where x and z are the coordinates on $\mathbb{R}^2$. Finally, Bazaikin has shown in [Baz09b, Thm. 2] that this metric is globally hyperbolic if ε is sufficiently small. □

Proposition 2.54. *Let (X, g^L, V) be an almost decent spacetime and $\mathcal{L}^\perp$ a leaf of $\mathcal{X}^\perp$. For any realization S of the screen bundle the transverse Levi-Civita connection of $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp}, g^R|_{\mathcal{L}^\perp})$ coincides with $\nabla^S|_{\mathcal{L}^\perp}$.*

Proof. Any local section $\tilde{V} \in \Gamma(U, \Xi)$ is given by $\tilde{V} = fV$. For $Y. \in \Gamma(U, S)$ and $\mathcal{F} := \mathcal{X}|_{\mathcal{L}^\perp}$ we compute

$$\nabla^T_{\tilde{V}} Y = \pi_{T\mathcal{F}^\perp}([\tilde{V}, Y]) = \pi_{T\mathcal{F}^\perp}(\nabla^L_{\tilde{V}} Y) - \pi_{T\mathcal{F}^\perp}(\underbrace{\nabla^L_Y fV}_{\in \Xi}) = \nabla^S_{\tilde{V}} Y.$$

The Koszul formula and the definition of g^R imply

$$\begin{aligned} 2g^R(\nabla^R_{Y_1} Y_2, Y_3) &= Y_1(g^R(Y_2, Y_3)) + Y_2(g^R(Y_1, Y_3)) - Y_3(g^R(Y_1, Y_2)) \\ &\quad + g^R([Y_1, Y_2], Y_3) - g^R([Y_1, Y_3], Y_2) - g^R([Y_2, Y_3], Y_1) \\ &= Y_1(g^L(Y_2, Y_3)) + Y_2(g^L(Y_1, Y_3)) - Y_L(g^R(Y_1, Y_2)) \\ &\quad + g^L([Y_1, Y_2], Y_3) - g^L([Y_1, Y_3], Y_2) - g^L([Y_2, Y_3], Y_1) \\ &= 2g^L(\nabla^L_{Y_1} Y_2, Y_3). \end{aligned}$$

Since $g^L(\nabla^S_{Y_1} Y_2, Y_3) = g^L(\nabla^L_{Y_1} Y_2, Y_3)$ we conclude the statement. $\square$

Suppose γ is a curve which is tangent to S such that $\gamma(0) \in \mathcal{L}^\perp$. Then the computations above imply that γ is a horizontal geodesic w.r.t. $g^R|_{\mathcal{L}^\perp}$ if $\nabla^S_{\dot\gamma}\dot\gamma = 0$.

Definition 2.55. *Let (X, g^L, V) be an almost decent spacetime. If S is a realization of the screen bundle we say (X, g^L, V, S) is*

- *almost horizontal if $\alpha(Y) = g^L(Z, \nabla^L_V Y)$ or equivalently $[V, Y] \in S$ for any local section $Y \in \Gamma(U, S)$,*
- *horizontal if it is almost horizontal and decent.*

Hence, $\nabla^L_V Y \in \Gamma(U, S)$ for any section $Y \in \Gamma(U, S)$ if (X, g^L, V, S) is horizontal. In particular, $d(g^L(Z, \cdot))(V, \cdot)|_{\Xi^\perp} = -g^L(Z, [V, \cdot])|_{\Xi^\perp} = 0$ if and only if (X, g^L, V, S) is almost horizontal. However, $d(g^L(Z, \cdot))(V, \cdot)|_{\Xi^\perp} = 0$ implies that $d(g^L(Z, \cdot))|_{\mathcal{L}^\perp}$ is a basic 2-form on $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ for any leaf $\mathcal{L}^\perp$ of $\mathcal{X}^\perp$.

Lemma 2.56. *Let (X, g^L, V) be an almost decent spacetime. If S is a realization of the screen bundle then*

1. *(X, g^L, V, S) is almost horizontal if and only if for any leaf $\mathcal{L}^\perp$ of $\mathcal{X}^\perp$ the restriction of $g^R|_{\mathcal{L}^\perp}$ of the (V, S)-metric defines the structure of an isometric Riemannian flow on $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$, i.e., $L_V g^R(W_1, W_2) = 0$ for all $W_1, W_2 \in \Xi^\perp$. Therefore, V is a $g^R|_{\mathcal{L}^\perp}$-Killing vector field of constant length.*
2. *The (V, S)-metric is bundle-like w.r.t. the foliation $(X, \mathcal{X})$ and $\alpha|_S = 0$ if and only if (X, g^L, V, S) is horizontal.*
3. *The (V, S)-metric g^R defines the structure of an isometric Riemannian flow on $(X, \mathcal{X})$ and $\alpha|_S = 0$ if and only if (X, g^L, V, S) is horizontal and $\alpha = 0$.*

Proof. Lemma 2.48 implies $L_V g^R(W_1, W_2) = 0$ for all $W_1, W_2 \in S$ and the first equivalence follows from

$$L_V g^R(V, W_2) = V(g^R(V, W_2)) - g^R([V, V], W_2) - g^R([V, W_2], V) = -g^L([V, W_2], Z).$$

If (X, g^L, V, S) is horizontal we have a local orthonormal frame $(Y_1, \dots, Y_{\dim S})$ for S such that $[V, Y_i] \in S$. Thus, $g^L(\nabla^L_V Z, Y_i) = -g^L(Z, \nabla^L_V Y_i) = 0$ and $pr_S([V, Z]) = pr_S(\nabla^L_V Z)$ imply $[V, Z] \in \Gamma(X, \Xi)$. This implies the second equivalence by Lemma 2.48. For the last statement we consider $L_V g^R(V, Z) = -g^R([V, Z], V) = \alpha(Z)$. $\square$

There exists a notion of *transverse conformal geometry* on horizontal spacetimes. More precisely, we have

Proposition 2.57. *Let (X, g^L, V, S) be a horizontal spacetime. If $f \in C^\infty(X)$ is $(X, \mathcal{X})$-basic, i.e., $V(f) = 0$, then (X, g^f) is a horizontal spacetime where the transverse conformal change g^f of g^L by f is defined by*

$$\begin{cases} g^f|_{S\times S} = e^f g^L|_{S\times S}, \\ g^f(V, V) = g^f(Z, Z) = g^f(V, S) = g^f(Z, S) = 0, \\ g^f(V, Z) = 1. \end{cases}$$

Proof. First, we show that $\nabla^f_\cdot V = \nabla^L_\cdot V$. Let $(V, Y_1, \dots, Y_{\dim S}, Z)$ be a local frame of (X, g^L) where $(Y_\alpha)_\alpha$ is a g^L-orthonormal frame for S. The Koszul formula and $\alpha|_{\Xi^\perp} = 0$ imply for $U_1, U_2 \in \{V, Y_\cdot, Z\}$

$$2g^f(\nabla^f_{U_1} V, U_2) = V g^f(U_1, U_2) + g^f([U_1, V], U_2) - g^f([V, U_2], U_1).$$

If $U_1 = U_2 = Z$ we derive $2g^f(\nabla^f_Z V, Z) = 2g^L(\nabla^L_Z V, Z) = 2\alpha(Z)$. If $U_1, U_2 \in \Xi^\perp$ we have $[V, U_\cdot] \in \Xi^\perp$. Hence,

$$\begin{aligned} 2g^f(\nabla^f_{U_1} V, U_2) &= V(e^f) g^L(U_1, U_2) + e^f(V g^L(U_1, U_2) + g^L([U_1, V], U_2) - g^L([V, U_2], U_1)) \\ &= V(e^f) g^L(U_1, U_2) + 2e^f g^L(\nabla^L_{U_1} V, U_2) = 0 \end{aligned}$$

since f is $(X, \mathcal{X})$-basic. Since (X, g^L) is horizontal we conclude

$$2g^f(\nabla^f_{U_1} V, U_2) = e^f g^L([Z, V], U_2) - g^L([V, U_2], Z) = (e^f - 1) g^L(Z, \nabla^L_V U_2) = 0$$

if $U_1 = Z$ and $U_2 \in S$. The case $U_1 \in S$ and $U_2 = Z$ is similar. On the other hand, $U_1 = V$ and $U_2 = Z$ implies $2g^f(\nabla^f_{U_1} V, U_2) = -g^L([V, Z], V) = \alpha(V) = 0$. Finally,

$$\begin{aligned} 2g^f(\nabla^f_V Y_\cdot, Z) &= g^f(\underbrace{[V, Y_\cdot]}_{\in S}, Z) - g^f([V, Z], Y_\cdot) - g^f([Y_\cdot, Z], V) \\ &= e^f g^L(Z, \underbrace{\nabla^L_V Y_\cdot}_{\in S}) + g^L(Z, \nabla^L_{Y_\cdot} V) = 0. \end{aligned}$$

Hence, (X, g^f, V, S) is horizontal. $\square$

If (X, g^L) is a Walker coordinate neighborhood of the form $g^L = 2dxdz + 2u_\alpha dy^\alpha dz + hdz^2 + g_{\alpha\beta}dy^\alpha dy^\beta$ and we choose $V := \partial_x$ and $Z := \partial_z - \frac{1}{2}h\partial_x$ then the transverse conformal change is given by $g^f = 2dxdz + 2u_\alpha dy^\alpha dz + hdz^2 + e^f g_{\alpha\beta}dy^\alpha dy^\beta$.

If (X, g^L, V, S) is horizontal then $[V, Z] \in \Gamma(X, \Xi)$, i.e., V and Z induce a 2-dimensional foliation on X. The (V, S)-metric g^R is bundle-like w.r.t. this foliation if $(L_Z g^R)|_{S\times S} = 0$. If (X, g^L) is a Walker coordinate neighborhood as above this condition corresponds to $\partial_z g_{\alpha\beta} = 0$.

Notice that Ex. 2.53 is in fact horizontal if $V = \partial_x$ and $S = TM$. More examples of horizontal spacetimes are discussed in [Lär10]. Here we only mention that another class of globally hyperbolic spacetimes was constructed in [BM08]. Using the notation of [BM08] we derive horizontal spacetimes if $S := TF$. Finally we provide a generalization of toric type manifolds which are clearly horizontal.

Proposition 2.58. *Let (M, h) be a Riemannian manifold admitting a global Killing vector field $V \in \Gamma(M, TM)$ such that $h(V, V) = 1$ and let $E :=$ span$\{V\}^\perp$. For $X := M \times \mathbb{R}$ write dz for the global coordinate 1-form on $\mathbb{R}$ and define*

$$g^L_f := 2h(V, \cdot)dz + fdz^2 + h(pr_E(\cdot), pr_E(\cdot))$$

for $f \in C^\infty(M)$. Then $(X., g^L_f)$ is a horizontal spacetime and V is g^L_f-parallel if f is a basic function w.r.t. the foliation induced by V.

Proof. Let $(Y_1, \dots, Y_n)$ be a local orthonormal frame of $(E, h|_E)$ and define $Z := \partial_z - \frac{1}{2}fV$ as well as $S := \text{span}\{V, Z\}^\perp = E$. If we write $\langle\cdot,\cdot\rangle := g^L_f$ the Koszul formula implies

$$\begin{aligned} 2\langle\nabla^L_{Y_k}V, Y_\ell\rangle &= \langle[Y_k, V], Y_\ell\rangle - \langle\underbrace{[Y_k, Y_\ell]}_{\in TM}, V\rangle - \langle[V, Y_\ell], Y_k\rangle \\ &= V(h(Y_k, Y_\ell)) - h([V, Y_k], Y_\ell) - h([V, Y_\ell], Y_k) \\ &= (L_V h)(Y_k, Y_\ell) = 0 \end{aligned}$$

and $2\langle\nabla^L_V V, Y_\ell\rangle = -2\langle\underbrace{[V, Y_\ell]}_{\in TM}, V\rangle = 0$. Moreover, $\langle\nabla^L_\cdot V, V\rangle = 0$ since V is lightlike and

$$\begin{aligned} 2\langle\nabla^L_Z V, Y_\ell\rangle &= \langle\underbrace{[Z, V]}_{=\frac{1}{2}V(f)V}, Y_\ell\rangle - \langle\underbrace{[Z, Y_\ell]}_{=\frac{1}{2}Y_\ell(f)V - \frac{1}{2}f[V, Y_\ell]}, V\rangle - \langle[V, Y_\ell], Z\rangle \\ &= -h([V, Y_\ell], V) = (L_V h)(V, Y_\ell) = 0. \end{aligned}$$

Hence, $\nabla^L_\cdot V = \alpha(\cdot)V$. In order to show that $\alpha|_{\Xi^\perp} = 0$ we compute

$$\begin{aligned} 2\langle\nabla^L_{Y_k}V, Z\rangle &= \langle[Y_k, V], Z\rangle - \langle\underbrace{[Y_k, Z]}_{\in TM}, V\rangle - \langle[V, Z], Y_k\rangle \\ &= h([Y_k, V], V) = (L_V h)(Y_k, V) = 0. \end{aligned}$$

Since $h([Y_k, V], V) = (L_V h)(Y_k, V) = 0$ we have $[Y_k, V] \in S$. Hence, (X, g^L_f, V, S) is horizontal. Finally,

$$2\langle \nabla^L_Z V, Z\rangle = 2\langle [Z, V], Z\rangle = \langle \frac{1}{2} V(f) V, Z\rangle = \frac{1}{2} V(f)$$

implies the last statement. $\square$

Proposition 2.59. *Let (X, g^L, V, S) be an almost horizontal spacetime and $\mathcal{L}^\perp$ a leaf of $\mathcal{X}^\perp$.*

1. *If all leaves of $\mathcal{X}|_{\mathcal{L}^\perp}$ are compact then the projection $\mathcal{L}^\perp \to \mathcal{L}^\perp/\mathcal{X}|_{\mathcal{L}^\perp}$ is a principal S^1-orbibundle over $\mathcal{L}^\perp/\mathcal{X}|_{\mathcal{L}^\perp}$ for which $S|_{\mathcal{L}^\perp}$ defines a connection whose connection 1-form is $g^L(Z, \cdot)|_{\mathcal{L}^\perp}$.*
2. *Furthermore, $\mathcal{L}^\perp \to \mathcal{L}^\perp/\mathcal{X}|_{\mathcal{L}^\perp}$ is a smooth principal S^1-bundle if each leaf of $\mathcal{X}|_{\mathcal{L}^\perp}$ has trivial leaf holonomy.*

Proof. The first statement is basically shown in [BG08, Thm. 6.3.8]. More precisely, $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp}, g^R|_{\mathcal{L}^\perp})$ defines an isometric Riemannian flow and a theorem of Wadsley [Wad75] implies that the leaves of $\mathcal{X}|_{\mathcal{L}^\perp}$ are the orbits of a smooth S^1-action. By Molino's Theorem 1.30 $\mathcal{L}^\perp \to \mathcal{L}^\perp/\mathcal{X}|_{\mathcal{L}^\perp}$ defines a Riemannian orbifold submersion, i.e., each leaf of $\mathcal{X}|_{\mathcal{L}^\perp}$ has finite leaf holonomy. Hence, the S^1-action is locally free, i.e., all isotropy groups are finite. However, as $\mathcal{X}|_{\mathcal{L}^\perp}$ is induced by an isometric S^1-action the isotropy groups are given by the leaf holonomy groups (to see this we may use a geodesic slice as in [BCO03, 3.1f]). Hence, $\mathcal{L}^\perp \to \mathcal{L}^\perp/\mathcal{X}|_{\mathcal{L}^\perp}$ is a principal S^1-bundle over the smooth manifold $\mathcal{L}^\perp/\mathcal{X}|_{\mathcal{L}^\perp}$ if all leaf holonomy groups of $\mathcal{X}|_{\mathcal{L}^\perp}$ vanish. $\square$

Remark 2.60. Notice that we used the almost horizontal property in Prop. 2.59 in order to derive the structure of an isometric Riemannian flow on $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$. However, we did not require $g^R|_{\mathcal{L}^\perp}$ to provide such a structure. Hence, the statements in Prop. 2.59 remain true if we replace horizontal by decent and require $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ to be taut. By a theorem of Ghys [Ghy84] this is the case if $\mathcal{L}^\perp$ is compact and simply connected. $\square$

Theorem 2.61. *Let (X, g^L, V, S) be a horizontal spacetime such that $Z \in \Gamma(X, TX)$ is complete. Suppose $\mathcal{L}^\perp$ is a leaf of $\mathcal{X}^\perp$ and write $\tilde{X}$ for the universal cover of X. If all leaves of $\mathcal{X}|_{\mathcal{L}^\perp}$ are compact with vanishing leaf holonomy then $\tilde{X}$ is diffeomorphic to the universal cover of a toric type Lorentzian manifold. Moreover, if $\mathcal{L}^\perp$ is closed in X then X is covered by a toric type Lorentzian manifold.*[18]

Proof. By Prop. 2.59 $\mathcal{L}^\perp$ is the total space of an S^1-bundle. Hence, there is a toric type metric on $\mathcal{L}^\perp \times \mathbb{R}$. However, Cor. 2.50 implies that $\tilde{X} = \tilde{\mathcal{L}^\perp} \times \mathbb{R}$ where $\tilde{\mathcal{L}^\perp}$ is the universal cover of $\mathcal{L}^\perp$. For the last statement Cor. 2.50.3 implies that $X \to X/\mathcal{X}^\perp$ is a fiber bundle over $X/\mathcal{X}^\perp \in \{\mathbb{R}, S^1\}$. Hence, X is covered by $\mathcal{L}^\perp \times \mathbb{R}$. $\square$

[18] In fact, Thm. 2.61 and Prop. 2.23 were the reason for the naming "horizontal".

In light of Prop. 2.59 we should stress the fact that the leaf holonomies of $\mathcal{X}|_{\mathcal{L}^\perp}$ are determined by the parallel displacement w.r.t. ∇^L in the following sense: If $\mathcal{L}$ is a leaf of $\mathcal{X}|_{\mathcal{L}^\perp}$ let γ be a loop in X such that $0 \neq [\gamma] \in \pi_1(\mathcal{L})$. If we fix a transversal $N_p \subset \mathcal{L}^\perp$ through $p := \gamma(0)$ then $Hol(\mathcal{L}, N_p)$ is defined (cf. Sec. 1.3) as a group of germs of diffeomorphisms of (N_p, p). On the other hand, we define the linearized holonomy group

$$Dhol(\mathcal{L}, N_p) := \{d\varphi|_p \in GL(\dim \mathcal{L}^\perp - 1, \mathbb{R}) : \varphi \in Hol(\mathcal{L}, N_p)\}.$$

Consider the transverse Levi-Civita connection ∇^T of $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp}, g^R|_{\mathcal{L}^\perp})$ and define the subgroup $H_p(\nabla^T) := \{\alpha \in Hol_p(\nabla^T) : \alpha \text{ is induced by a loop } \gamma \subset \mathcal{L}\}$. It is shown in [Mor76, Thm. 2] that $H_p(\nabla^T) = Dhol(\mathcal{L}, N_p)$.[19] Since $g^R|_{\mathcal{L}^\perp}$ is bundle-like w.r.t. $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ the leaf holonomy group is a group of germs of isometric diffeomorphisms of $((N_p, p), g^R|_{N_p})$, i.e, if $Dhol(\mathcal{L}, N_p)$ is trivial so is $Hol(\mathcal{L}, N_p)$. Hence, Prop. 2.54 implies

Corollary 2.62. *Let (X, g^L, V) be an almost decent spacetime and $\mathcal{L}^\perp$ a leaf of $\mathcal{X}^\perp$. If $\mathcal{L}$ is a leaf of $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ then $Hol_p(\mathcal{L})$ is trivial if and only if*

$$\{\tau_\gamma^{\nabla^S} : \gamma \subset \mathcal{L} \textit{ is a loop based at } p\} = \{id\}.$$

□

Given Prop. 2.59 it is natural to pose the question as to whether $\mathcal{L}^\perp$ arises as a principal $\mathbb{R}$-bundle if (X, g^L) is a strongly causal spacetime.

We remind that for a decent spacetime (X, g^L, V) the integral curves of V are lightlike geodesics, i.e., if (X, g^L) is lightlike complete so is V.

Theorem 2.63. *Let (X, g^L, V) be a causal, almost decent spacetime such that V is complete and let S be a realization of the screen bundle with its (V, S)-metric g^R. Suppose $\mathcal{L}^\perp$ is a leaf of $\mathcal{X}^\perp$ through $p \in X$ and one of the following conditions holds:*

- *$g^R|_{\mathcal{L}^\perp}$ is complete and (X, g^L) is strongly causal at $p \in X$,*
- *The quotient topology on $M := \mathcal{L}^\perp / \mathcal{X}|_{\mathcal{L}^\perp}$ is Hausdorff.*

Then M is a smooth manifold and $\pi : \mathcal{L}^\perp \to M$ is a smooth principal $\mathbb{R}$-bundle. In particular, $\mathcal{L}^\perp = \mathbb{R} \times M$ and $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ is taut.

Proof. Consider the Riemannian flow $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$. We have shown in Prop. 2.52 that all leaves of $\mathcal{X}|_{\mathcal{L}^\perp}$ have trivial leaf holonomy and vanishing fundamental group if (X, g^L) is causal. It is well known that π is a submersion if M is Hausdorff (cf. [Sha97, Thm. 8.3]). On the other hand, if (X, g^L) is strongly causal at $p \in X$ then Prop. 2.52 implies that $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ is regular at p. Moreover, if $g^R|_{\mathcal{L}^\perp}$ is complete then $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ is regular for all $q \in \mathcal{L}^\perp$ (cf. [Rei59, Lemma 6]) since all leaves have trivial leaf holonomy. In this case, all leaves of $\mathcal{X}|_{\mathcal{L}^\perp}$ are closed in $\mathcal{L}^\perp$ (cf. [Rei59, Cor. 3]) and M is Hausdorff by Thm. 1.29.

[19] In fact, the statement is shown for any basic connection. Note, that a basic connection is called Bott connection in [Mor76].

If V is complete its flow ψ_t induces a smooth $\mathbb{R}$-action $\psi : \mathcal{L}^\perp \times \mathbb{R} \to \mathcal{L}^\perp$ whose orbits are precisely the leaves of $\mathcal{X}|_{\mathcal{L}^\perp}$. Suppose we have already shown that ψ makes $\pi : \mathcal{L}^\perp \to M$ a smooth principal $\mathbb{R}$-bundle. Then $\mathcal{L}^\perp = \mathbb{R} \times M$. More precisely, using the Čech-description of principal bundles the set of isomorphism classes of principal $\mathbb{R}$-bundles over M is given by $\check{H}^1(M, \mathcal{C}^\infty_{\mathbb{R}})$ which is trivial as $\mathcal{C}^\infty_{\mathbb{R}}$ is a fine sheaf.

Suppose ψ does not induce a free $\mathbb{R}$-action. Then there exists $p \in \mathcal{L}^\perp$ and $t_0, t_1 \in \mathbb{R}$ such that $\psi_{t_0}(p) = \psi_{t_1}(p)$, i.e., $\gamma : [t_0, t_1] \to \mathcal{L}^\perp$ defined by $\gamma(t) := \psi_t(p)$ is a closed lightlike curve in X and (X, g^L) is not causal.

Hence, we have a Lie group G acting smoothly and freely on a manifold X such that $\pi : X \to M := X/_G$ is a smooth submersion and it is well known that π is a smooth principal G-bundle in that case. This fact can be seen as follows:

Since π is a submersion there is a local section $\sigma : U \subset M \to X$. If we write $\psi : X \times G \to X$ for the G-action then $F : U \times G \to \pi^{-1}(U)$ such that $F(p, g) \mapsto \psi(\sigma(p), g)$ is smooth and surjective. Since G acts freely F is injective and the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xleftarrow{F} & U \times G \\ {\scriptstyle\pi}\searrow & & \swarrow {\scriptstyle pr_1} \\ & U & \end{array}$$

commutes. If we let G act on $U \times G$ from the right then F is G-equivariant and all we need to show is that F is an immersion. However, using the diagram we conclude $Ker(F_*) \subset \{0\} \times TG$. Moreover, any orbit map $g \mapsto p \cdot g$ and therefore F is an immersion, i.e., F is a diffeomorphism. Thus, π is a principal G-bundle.

In particular, G acts properly on X.[20] However, given a smooth, free, proper action of G on X there is a G-invariant Riemannian metric g on X (cf. [BG08, Thm. 1.6.17]). Hence, $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ is taut. $\square$

Proposition 2.64. *Let (X, g^L, V, S) be a strongly causal, almost horizontal spacetime such that V is complete. For any leaf $\mathcal{L}^\perp$ of $\mathcal{X}^\perp$ the map $\pi : \mathcal{L}^\perp \to M := \mathcal{L}^\perp/\mathcal{X}|_{\mathcal{L}^\perp}$ is a smooth principal $\mathbb{R}$-bundle over M, i.e, $\mathcal{L}^\perp = \mathbb{R} \times M$.*

Proof. By Thm. 2.63 we need to prove that M is Hausdorff and this seems to be known for isometric flows. Since I could not find a reference including a proof we have to suffer once more.

Let $p, q \in \mathcal{L}^\perp$ such that $\pi(p) \neq \pi(q)$ and write $\mathcal{L}_p$ for the leaf of $\mathcal{X}|_{\mathcal{L}^\perp}$ through p. As above the flow ψ of V induces a smooth, free $\mathbb{R}$-action. Since (X, g^L, V, S) is almost horizontal ψ induces a $g^R|_{\mathcal{L}^\perp}$-isometric action by Lemma 2.56. Let $\varepsilon > 0$ and define $B_\varepsilon(p) := \{q \in \mathcal{L}^\perp : d(p, q) < \varepsilon\}$ where d is the distance function induced by $g^R|_{\mathcal{L}^\perp}$.

Since ψ is an isometric action we conclude $\psi_t(B_\varepsilon(p)) = B_\varepsilon(\psi_t(p))$ for all $t \in \mathbb{R}$ and any $p \in \mathcal{L}^\perp$. Moreover,

$$\pi^{-1}(\pi(B_\varepsilon(p))) = \mathbb{R} \cdot B_\varepsilon(p) = \psi_{\mathbb{R}}(B_\varepsilon(p)) = \bigcup_{t \in \mathbb{R}} \psi_t(B_\varepsilon(p)) = \bigcup_{t \in \mathbb{R}} B_\varepsilon(\psi_t(p)).$$

[20] We say that a smooth G-action of a Lie group G on the manifold X is proper if the map $G \times X \to X \times X$ such that $(g, p) \mapsto (gp, p)$ is proper.

Since $\bigcup_{t\in\mathbb{R}} B_\varepsilon(\psi_t(p))$ is open in $\mathcal{L}^\perp$ the subset $\pi(B_\varepsilon(p))$ is open in M with the quotient topology. Since all leaves of $\mathcal{X}|_{\mathcal{L}^\perp}$ are closed in $\mathcal{L}^\perp$ there is $\varepsilon > 0$ such that $B_\varepsilon(p)\cap\mathcal{L}_q = \emptyset$. Thus, $\pi(B_{\frac{\varepsilon}{2}}(p)), \pi(B_{\frac{\varepsilon}{2}}(q))$ are disjoint and open in M. To see this, suppose there is $x \in \pi^{-1}(\pi(B_\varepsilon(p)))\cap\pi^{-1}(\pi(B_\varepsilon(q)))$. Then there is $\tilde{p} := \psi_{\tilde{t}}(p) \in \mathcal{L}_p$ and $\tilde{q} \in \mathcal{L}_q$ such that $d(x,\tilde{p}), d(x,\tilde{q}) < \frac{\varepsilon}{2}$. Since ψ is isometric we derive $d(p,\psi_{-\tilde{t}}(\tilde{q})) = d(\tilde{p},\tilde{q}) < \varepsilon$ implying the contradiction $\psi_{-\tilde{t}}(\tilde{q}) \in \mathcal{L}_q \cap B_\varepsilon(p)$. □

Remark 2.65. Notice that we can apply the same proof if we replace almost horizontal by almost decent in Prop. 2.64 and assume $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ to be taut. In this case, we have to replace $g^R|_{\mathcal{L}^\perp}$ by a Riemannian metric making V a Killing vector field of unit length. □

For the next statement we remind that all integral curves of $Z \in \Gamma(X, TX)$ are g^R-geodesics.

Theorem 2.66. *Let (X, g^L, V) be a simply connected, causal, decent spacetime such that V is complete. Let $\mathcal{L}^\perp$ be any leaf of $\mathcal{X}^\perp$ and S a realization of the screen bundle. Suppose $Z \in \Gamma(X, TX)$ is complete and one of the following conditions holds:*

- *$g^R|_{\mathcal{L}^\perp}$ is complete and (X, g^L) is strongly causal at some $p \in X$,*
- *(X, g^L, V, S) is horizontal and strongly causal.*

Then $\mathcal{L}^\perp = \mathbb{R}\times M$ and $X = \mathbb{R}^2 \times M$ where $M := \mathcal{L}^\perp/\mathcal{X}|_{\mathcal{L}^\perp}$ is a smooth manifold.

Proof. Since Z is complete and X is simply connected Cor. 2.50 implies $X = \mathcal{L}^\perp \times \mathbb{R}$. The other conditions imply $\mathcal{L}^\perp = \mathbb{R}\times M$ by Thm. 2.63 resp. Prop. 2.64. □

Suppose (X, g^L, V) is as in Thm. 2.66 such that $g^R|_{\mathcal{L}^\perp}$ is complete. By Thm. 1.29 we derive a complete Riemannian metric h on M making $\pi : (\mathcal{L}^\perp, g^R|_{\mathcal{L}^\perp}) \to M$ a Riemannian submersion. In particular, any loop in M admits a horizontal lift. Hence, $Hol(M,h) \subset Hol(\nabla^T)$ and Prop. 2.54 implies $Hol(M,h) \subset Hol(\nabla^S|_{\mathcal{L}^\perp})$.

Corollary 2.67. *Let (X, g^L, V) be a simply connected, lightlike complete, causal, decent spacetime such that $\dim_{\mathbb{R}} X = 10$ and $b_6(X) = 1$. Let $\mathcal{L}^\perp$ be a leaf of $\mathcal{X}^\perp$ and suppose there is a realization S of the screen bundle such that $g^R|_{\mathcal{L}^\perp}$ and $Z \in \Gamma(X,TX)$ are complete.*

If $Hol(\nabla^S|_{\mathcal{L}^\perp}) \subset \{0\}\times SU(3)$ then $X = \mathbb{R}^4 \times M$ where M is a simply connected compact manifold admitting a Ricci-flat Kähler metric.

Proof. Thm. 2.66 implies that $N := \mathcal{L}^\perp/\mathcal{X}|_{\mathcal{L}^\perp}$ is smooth and $X = \mathbb{R}^2 \times N$. As we have explained above there is a complete Riemannian metric h on N such that $Hol(N,h) \subset \{0\}\times SU(3)$. The de Rham decomposition theorem implies $N = \mathbb{R}^2 \times M$ where M admits a metric whose holonomy is contained in $SU(3)$. Hence, there is a Ricci-flat Kähler metric on M. Finally, M is compact as $b_6(X) = b_{\dim_{\mathbb{R}} M}(M) = 1$. □

Remark 2.68. Thm. 2.66 provides sufficient conditions for a strongly causal, decent spacetime to be diffeomorphic to $\mathbb{R}^2 \times M$. On the other hand, sufficient conditions for $(\mathbb{R}^2 \times M, 2dxdz + fdz^2 + g_M)$ to be strongly causal have been found in [FS03]. □

Any stably causal almost decent spacetime admits an integrable realization of the screen bundle by Prop. 2.52. Hence, we review spacetimes with an integrable realization of the screen bundle for which the main tool is given by

Theorem 2.69 (Blumenthal-Hebda decomposition [BH83])**.** *Let (X, g) be a complete Riemannian manifold and $\mathcal{F}$ a totally geodesic foliation on X whose normal bundle $T\mathcal{F}^\perp$ is integrable. Then $\tilde{X}$ is diffeomorphic to $\tilde{L} \times \tilde{H}$ where $\tilde{X}$ is the universal cover of X and $\tilde{L}$ resp. $\tilde{H}$ is the universal cover of a leaf of $\mathcal{F}$ resp. $T\mathcal{F}^\perp$.* □

Corollary 2.70. *Let (X, g^L, V) be an almost decent spacetime and $\mathcal{L}^\perp$ a leaf of $\mathcal{X}^\perp$. Suppose S is an integrable realization of the screen bundle and write $\tilde{\mathcal{L}}^\perp$ for the universal cover of $\mathcal{L}^\perp$.*

1. *If $g^R|_{\mathcal{L}^\perp}$ is complete then $\tilde{\mathcal{L}}^\perp = \mathbb{R} \times \tilde{\mathcal{S}}$ where $\tilde{\mathcal{S}}$ is the universal cover of a leaf of $S|_{\mathcal{L}^\perp}$.*
2. *If (X, g^L, V, S) is horizontal such that $(L_Z g^L)|_{S\times S} = 0$ and g^R is complete then $\tilde{X} = \mathbb{R}^2 \times \tilde{\mathcal{S}}$ where $\tilde{\mathcal{S}}$ is a leaf of $S|_{\mathcal{L}^\perp}$.*
3. *In both cases, if $Hol^0(\nabla^S) \subset H_1 \times H_2$ then $\tilde{\mathcal{S}} = \tilde{\mathcal{S}}_1 \times \tilde{\mathcal{S}}_2$ as Riemannian manifolds and $Hol(\tilde{\mathcal{S}}_i) \subset H_i$.*

Proof. Since $g^R|_{\mathcal{L}^\perp}$ is bundle-like for $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ the leaves of $S|_{\mathcal{L}^\perp}$ are totally geodesic in $\mathcal{L}^\perp$ and we can apply Thm. 2.69.

As we have seen in the discussion following Prop. 2.57 the vector fields V and Z induce a 2-dimensional foliation on X if (X, g^L, V, S) is horizontal. In this case, g^R is bundle-like w.r.t. this foliation if $(L_Z g^L)|_{S\times S} = 0$. Thus, S is totally geodesic in (X, g^R) and Thm. 2.69 implies $\tilde{X} = M \times \tilde{\mathcal{S}}$ where M is the universal cover of a leaf of the foliation induced by V and Z. Since M is a simply connected parallelizable surface the uniformization theorem implies $M \cong \mathbb{R}^2$. The last statement follows from the geometric de Rham decomposition theorem since $Hol(\nabla^S|_{\tilde{\mathcal{S}}}) \subset Hol(\nabla^S|_{\tilde{\mathcal{L}}^\perp}) \subset Hol(\nabla^S)$. □

Suppose M is a compact simply connected manifold and $\tilde{X} \to M$ is an S^1-bundle such that $0 \neq c_1(\tilde{X} \to M) \in H^2(M, \mathbb{Z})$. Gysin's sequence implies

$$0 = H^1(\tilde{X}, \mathbb{Z}) \to H^1(\tilde{X}, \mathbb{Z}) \to H^0(M, \mathbb{Z}) \xrightarrow{c_1} H^2(M, \mathbb{Z}),$$

i.e., $H^1(\tilde{X}, \mathbb{Z}) = 0$. By Serre's sequence we have $\pi_2(M) \to \mathbb{Z} \to \pi_1(\tilde{X}) \to 0$, i.e., $\pi_1(\tilde{X}) = H_1(\tilde{X}, \mathbb{Z})$ is abelian. Since $H^1(\tilde{X}, \mathbb{Z})$ is finite the universal coefficient theorem implies that $H_1(\tilde{X}, \mathbb{Z})$ is a finite torsion group. Hence, the universal cover of $\tilde{X}$ is compact.

Suppose $(X = \tilde{X} \times L, \tilde{g}_f)$ is of toric type, i.e., $\tilde{X} = \mathcal{L}^\perp$. Then $\tilde{\mathcal{L}}^\perp$ is compact and Cor. 2.70 implies

Corollary 2.71. *Let (M, g) be a compact simply connected Riemannian manifold and let $0 \neq [\frac{\psi}{2\pi}] \in H^2(M, \mathbb{Z})$. Suppose $(X = \tilde{X} \times L, \tilde{g}_f)$ is of toric type where $\tilde{X} \to M$ is the S^1-bundle corresponding to $-[\frac{\psi}{2\pi}]$. Then $(X, \tilde{g}_f)$ does not admit an integrable realization of the screen bundle.* □

Remark 2.72. Let (S^2, g) be the round sphere and $0 \neq [\frac{\psi}{2\pi}] \in H^2(S^2, \mathbb{Z})$. If $(X = \tilde{X} \times L, \tilde{g}_f)$ is of toric type where $\tilde{X} \to S^2$ is the S^1-bundle corresponding to $-[\frac{\psi}{2\pi}]$ then the Milnor-Wood inequality (cf. [Woo71, Thm. 1.1]) implies that $\mathcal{L}^\perp = \tilde{X}$ does not admit a foliation whose leaves are transverse to $\mathcal{X}|_{\mathcal{L}^\perp}$. □

Finally, we study holonomy representations of decent spacetimes.

By Prop. 2.54 there is a (torsion-free) transverse G-structure in the sense of Section 1.3 on $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ if $Hol(\nabla^S|_{\mathcal{L}^\perp}) \subset G$. We have already discussed below Lemma 2.14 that the inclusion $Hol(\nabla^S|_{\mathcal{L}^\perp}) \subset Hol(\nabla^S)$ can be strict. Since $\mathfrak{hol}(\nabla^S)$ has the Borel-Lichnérowicz property by Cor. 2.2 there are decompositions $S_p = E_0 \oplus \ldots \oplus E_\ell$ and $Hol^0_p(\nabla^S) = H_1 \oplus \ldots \oplus H_\ell$, where H_j acts irreducibly on E_j for $j \geq 1$.

If $\gamma : [0,1] \to X$ is a piecewise smooth curve such that $\gamma(0) = p$ and if τ^S_γ is the parallel displacement w.r.t. ∇^S along γ we define $R_p^{\tau^S_\gamma}(v, w) := {\tau^S_\gamma}^{-1} \circ R^S_{\gamma(1)}(w, v) \circ \tau^S_\gamma$ for $v, w \in S_{\gamma(1)}$. The Ambrose-Singer theorem and $R^S(V, \Xi^\perp) = 0$ imply

$$\mathfrak{hol}_p(\nabla^S|_{\mathcal{L}^\perp}) = \mathrm{span}\{R_p^{\tau^S_\gamma}(\tau^S_\gamma v, \tau^S_\gamma w) : v, w \in S_p, \gamma : [0,1] \to \mathcal{L}^\perp\}.$$

Moreover, each $R_p^{\tau^S_\gamma}(\tau^S_\gamma(\cdot), \tau^S_\gamma(\cdot))$ is an algebraic curvature tensor on S_p. Therefore, $\mathfrak{hol}_p(\nabla^S|_{\mathcal{L}^\perp})$ is a Berger algebra in $\mathfrak{so}(S_p)$, i.e., it acts as a Riemannian holonomy representation. Since each subspace E_j is $Hol^0_p(\nabla^S|_{\mathcal{L}^\perp})$-invariant we may consider

$$\mathcal{K}(E_j) := \mathrm{span}\{R_p^{\tau^S_\gamma}(\tau^S_\gamma(\cdot), \tau^S_\gamma(\cdot))|_{E_j \times E_j \times E_j}\}.$$

Suppose $0 \neq \tilde{R} \in \mathcal{K}(E_k)$. Then $(E_k, \tilde{R}, H_k)$ is an irreducible holonomy system and Simons' theorem [Sim62] implies that H_k acts on E_k as a Riemannian holonomy representation. Hence, Leistner's classification (Thm. 1.9) is only necessary for those H_k for which $\mathcal{K}(E_k) = 0$.

Lemma 2.73. *Let (X, g^L, V) be an almost decent spacetime and S a realization of the screen bundle. Suppose there is a leaf $\mathcal{L}^\perp$ of $\mathcal{X}^\perp$ such that $(\mathcal{L}^\perp, g^R|_{\mathcal{L}^\perp})$ is complete. If $p \in \mathcal{L}^\perp$ and $\mathcal{K}(E_k) = 0$ then $\tilde{\mathcal{L}}^\perp = A \times \mathbb{R}^{\dim E_k}$ where $\tilde{\mathcal{L}}^\perp$ is the universal cover of $\mathcal{L}^\perp$.*

Proof. Consider the foliated manifold $(\tilde{\mathcal{L}}^\perp, \tilde{\mathcal{X}}|_{\mathcal{L}^\perp}, \tilde{g}^R|_{\mathcal{L}^\perp})$ together with the lifted connection $\tilde{\nabla}^S|_{\mathcal{L}^\perp}$. Since $\tilde{\mathcal{L}}^\perp$ is simply connected we have $\tilde{\nabla}^S|_{\mathcal{L}^\perp}$-parallel orthonormal sections $Y_1, \ldots, Y_{\dim E_k} \in \Gamma(\tilde{\mathcal{L}}^\perp, \tilde{S})$. An integral curve of any Y_i is a horizontal $\tilde{g}^R|_{\mathcal{L}^\perp}$-geodesic. Hence, each Y_i is a complete vector field on $\tilde{\mathcal{L}}^\perp$. Define $\mathcal{T}^{Y_1} := \mathrm{span}\{Y_1\}^\perp \subset T\tilde{\mathcal{L}}^\perp$. If $W \in \Gamma(U, \mathcal{T}^{Y_1})$ is a local section then $[W, Y_1] \in \tilde{\nabla}^S_W Y_1 - \tilde{\nabla}^S_{Y_1} W + \tilde{\mathcal{X}}|_{\mathcal{L}^\perp} \subset \mathcal{T}^{Y_1}$. Moreover, $\tilde{\nabla}^S_\cdot \mathcal{T}^{Y_1} \subset \mathcal{T}^{Y_1}$.

Thus, $\mathcal{T}^{Y_1}$ induces a transversely parallelizable codimension one foliation in $\mathcal{L}^\perp$ and Prop. 1.38.3 implies $\mathcal{L}^\perp = A_{Y_1} \times \mathbb{R}$ where A_{Y_1} is a leaf of $\mathcal{T}^{Y_1}$. For $i \geq 2$ we restrict the vector fields Y_i to A_{Y_1}. As above, we derive a transversely parallelizable codimension one foliation on A_{Y_1} induced by $\mathcal{T}^{Y_2} := \mathrm{span}\{Y_2|_{A_{Y_1}}\}^\perp$ and Y_2 is a complete transverse vector field. Inductively, we have $\tilde{\mathcal{L}}^\perp = A \times \mathbb{R}^{\dim E_k}$. □

Theorem 2.74. *Let (X, g^L) be a time-orientable Lorentzian manifold where $\mathfrak{hol}(X, g^L)$ acts weakly irreducibly with index 1 and suppose the associated foliation $\mathcal{X}^\perp$ admits a compact leaf $\mathcal{L}^\perp$ such that $\pi_1(\mathcal{L}^\perp)$ is finite. Then $\mathfrak{hol}(X, g^L)$ belongs to one of the following types where $\mathfrak{g} := \mathfrak{hol}(\nabla^S)$.*

- *Type 1:* $\mathfrak{hol}(X, g^L) = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^{\dim X - 2}$
- *Type 2:* $\mathfrak{hol}(X, g^L) = \mathfrak{g} \ltimes \mathbb{R}^{\dim X - 2}$
- *Type 3:*

$$\mathfrak{hol}(X, g^L) = \left\{ \begin{pmatrix} \varphi(A) & w^T & 0 \\ 0 & A & -w \\ 0 & 0 & -\varphi(A) \end{pmatrix} : A \in \mathfrak{g},\ w \in \mathbb{R}^{\dim X - 2} \right\}$$

where $\varphi : \mathfrak{g} \twoheadrightarrow \mathbb{R}$ is an epimorphism satisfying $\varphi|_{[\mathfrak{g},\mathfrak{g}]} = 0$.

Moreover, identifying $\mathfrak{g} \subset \mathfrak{so}(\dim X - 2)$ there are decompositions $\mathbb{R}^{\dim X - 2} = F_1 \oplus \ldots \oplus F_\ell$ and $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_\ell$ such that each $\mathfrak{g}_j$ acts trivially on F_i for $i \neq j$ and as an irreducible Riemannian holonomy representation on F_j. In particular, $\mathfrak{g}$ does not act trivially on any subspace of $\mathbb{R}^{\dim X - 2}$.

Proof. The universal cover of $\mathcal{L}^\perp$ is compact, i.e., $\mathfrak{g}$ does not act trivially on any subspace of $\mathbb{R}^{\dim X - 2}$ by Lemma 2.73. Hence, $\mathcal{K}(F_k) \neq 0$ for all k and the statement for $\mathfrak{g}$ follows from the discussion above. By Thm. 2.11 $\mathfrak{hol}(X, g^L)$ does not belong to one of the three types if it is given as follows: There is $0 < \ell < \dim X - 2$ such that $\mathbb{R}^{\dim X - 2} = \mathbb{R}^\ell \oplus \mathbb{R}^{\dim X - 2 - \ell}$, $\mathfrak{g} \subset \mathfrak{so}(\ell)$ and

$$\mathfrak{hol}(X, g^L) = \left\{ \begin{pmatrix} 0 & \psi(A)^T & w^T & 0 \\ 0 & 0 & 0 & -\psi(A) \\ 0 & 0 & A & -w \\ 0 & 0 & 0 & 0 \end{pmatrix} : A \in \mathfrak{g},\ w \in \mathbb{R}^\ell \right\}$$

for some epimorphism $\psi : \mathfrak{g} \twoheadrightarrow \mathbb{R}^{\dim X - 2 - \ell}$ satisfying $\psi|_{[\mathfrak{g},\mathfrak{g}]} = 0$. However, in this case $\mathfrak{g}$ would act trivially on $\mathbb{R}^{\dim X - 2 - \ell}$. □

The key idea for the construction of examples of toric type manifolds $(X = \tilde{X} \times L, \tilde{g}_f)$ such that $Hol(\nabla^S) \subset G$ was given in Prop. 2.25. More precisely, we constructed $\tilde{X}$ over a Riemannian manifold (M, g) such that $Hol(M, g) \subset G$ and considered the lift of a $Hol(M, g)$-invariant tensor A to S. The remaining condition $\nabla^S_Z A = 0$ implied a relation on the Euler class of the S^1-bundle $\tilde{X} \to M$ which we solved using Hodge theory on M.

Now, consider the general case, i.e., let A be a global section of some tensor bundle of S and suppose that $A|_{\mathcal{L}^\perp}$ is invariant under the action of $Hol(\nabla^S|_{\mathcal{L}^\perp})$ for any leaf $\mathcal{L}^\perp$ of $\mathcal{X}^\perp$. Then A is invariant under the action of $Hol(\nabla^S)$ if and only if $\nabla^S_Z A = 0$. By Prop. 2.54 $\nabla^S|_{\mathcal{L}^\perp} A|_{\mathcal{L}^\perp}$ means that $A|_{\mathcal{L}^\perp}$ induces a (torsion-free) transverse $\mathrm{Stab}_{O(S)}\{A\}$-structure on $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$.

Lemma 2.75. *Let (X, g^L, V) be an almost decent spacetime and S a realization of the screen bundle. If $J \in \Gamma(X, O(S))$ with $J^2 = -id_S$ then $\nabla^S J = 0$ if and only if $\nabla^S|_{\mathcal{L}^\perp} J|_{\mathcal{L}^\perp} = 0$ for any leaf $\mathcal{L}^\perp$ of $\mathcal{X}^\perp$ and*

$$\begin{aligned} 0 &= d(g^L(Z,\cdot))(JY_1, Y_2) + d(g^L(Z,\cdot))(Y_1, JY_2) \\ &\quad + g^L((L_Z J)(Y_1), Y_2) - g^L((L_Z J)(Y_2), Y_1). \end{aligned}$$

Proof. Define the extension $J \in \Gamma(X, End(TX))$ by $J(V) = J(Z) = 0$ and let $\omega(\cdot,\cdot) := g^L(J(\cdot),\cdot) \in \Lambda^2 T^* X$. Since $(L_Z J)(Y) = [Z, JY] - J([Z, Y])$ we compute for $Y_1, Y_2 \in \Gamma(U, S)$

$$\begin{aligned} g^L((\nabla^S_Z J)(Y_1), Y_2) &= g^L(\nabla^S_Z (JY_1), Y_2) - g^L(J\nabla^S_Z Y_1, Y_2) \\ &= g^L([Z, JY_1], Y_2) + g^L(\nabla^L_{JY_1} Z, Y_2) \\ &\quad + g^L([Z, Y_1], JY_2) + g^L(\nabla^L_{Y_1} Z, JY_2) \\ &= g^L((L_Z J)(Y_1), Y_2) + g^L(\nabla^L_{JY_1} Z, Y_2) + g^L(\nabla^L_{Y_1} Z, JY_2). \end{aligned}$$

Therefore,

$$\begin{aligned} & g^L((\nabla^S_Z J)(Y_1), Y_2) - g^L((\nabla^S_Z J)(Y_2), Y_1) \\ & = g^L((L_Z J)(Y_1), Y_2) - g^L((L_Z J)(Y_2), Y_1) + d(g^L(Z,\cdot))(JY_1, Y_2) + d(g^L(Z,\cdot))(Y_1, JY_2) \end{aligned}$$

and we conclude the statement follows since $\nabla^S_\cdot \omega$ is a 2-form on S and $\nabla^S_Z \omega(Y_1, Y_2) = g^L((\nabla^S_Z J)(Y_1), Y_2)$. □

Remark 2.78 will show that $d(g^L(Z,\cdot))|_{\mathcal{L}^\perp}$ induces the Euler class of the Riemannian flow $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ if (X, g^L, V, S) is almost horizontal. In that case, $\nabla^S J = 0$ and $L_Z J = 0$ implies that $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ is a transverse Kähler foliations satisfying the condition $[d(g^L(Z,\cdot))|_{\mathcal{L}^\perp}] \in H^{1,1}_B(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$, where $H^{p,q}_B$ is the basic Dolbeault cohomology of $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp}, J|_{\mathcal{L}^\perp})$.[21] If (X, g^L, V, S) is horizontal such that $L_Z J = 0$ we have the following application.

Proposition 2.76. *Let (X, g^L, V, S) be a horizontal spacetime and $\mathcal{L}^\perp$ a leaf of $\mathcal{X}^\perp$. Suppose $d(g^L(Z,\cdot))|_{\mathcal{L}^\perp} \in \Lambda^{1,1}_B \mathcal{X}|_{\mathcal{L}^\perp}$ for some ∇^S-parallel almost Hermitian structure J on S. If $Z \in \Gamma(X, TX)$ is complete then there exists a complex structure on the universal cover of X.*

Proof. By Cor. 2.50 the universal cover of X is diffeomorphic to $\tilde{X} := \mathcal{L}^\perp \times \mathbb{R}^+$. We write r for the coordinate on $\mathbb{R}^+$ and $\eta := g^L(Z,\cdot)|_{T\mathcal{L}^\perp}$. If $\Phi \in \mathrm{End}(T\mathcal{L}^\perp)$ is given by $\Phi(w \in S_p) := J(w)$ and $\Phi(V) := 0$ then $(V, \eta, \Phi, g^R|_{\mathcal{L}^\perp})$ defines an almost contact metric structure on $\mathcal{L}^\perp$. On $\tilde{X}$ we define the cone metric $g^C := dr^r + r^2 g^R|_{\mathcal{L}^\perp}$ and the section

[21] Basic (p,q)-forms are defined in the same way as we did for almost complex manifolds. In the next section we come back to this point and refer to [EKA90] for the moment.

$I \in \mathrm{End}(T\tilde{X})$ by

$$IY := \begin{cases} JY & \text{if } Y \in S_p, \\ r\partial_r & \text{if } Y = V, \\ -V & \text{if } Y = r\partial_r. \end{cases}$$

Hence, we derive an almost Hermitian manifold $(\tilde{X}, I, g^C)$. By [BG08, Thm. 6.5.9] I is integrable once we prove[22] that $N_\Phi = -V \otimes d\eta$ where $N_\Phi(Y_1, Y_2) := [\Phi Y_1, \Phi Y_2] + \Phi^2[Y_1, Y_2] - \Phi[Y_1, \Phi Y_2] - \Phi[\Phi Y_1, Y_2]$ for $Y_\cdot \in T\mathcal{L}^\perp$. Let $Y_1 \in S$ and $Y_2 = V$. Since (X, g^L, V, S) is horizontal we have $[Y_1, V] = -\nabla^S_V Y_1$. Thus, $\Phi V = 0$ and $J \circ \nabla^S = \nabla^S \circ J$ implies

$$\begin{aligned} N_\Phi(Y_1, V) &= \Phi^2[Y_1, V] - \Phi[\Phi Y_1, V] = -\Phi^2(\nabla^S_V Y_1) + \Phi(\nabla^S_V \Phi Y_1) \\ &= -J^2(\nabla^S_V Y_1) + J(\nabla^S_V J Y_1) = 0. \end{aligned}$$

The same way we compute $g^L(N_\Phi(Y_1, Y_2), Y_3) = 0$ if $Y_1, Y_2, Y_3 \in S$. For $Y_1, Y_2 \in S$ we have

$$\begin{aligned} g^L(N_\Phi(Y_1, Y_2), Z) &= g^L([\Phi Y_1, \Phi Y_2], Z) = g^L([JY_1, JY_2], Z) \\ &= -g^L(\nabla^L_{JY_1} Z, JY_2) + g^L(\nabla^L_{JY_2} Z, JY_1) \\ &= -d(g^L(Z, \cdot))|_{\mathcal{L}^\perp}(JY_1, JY_2). \end{aligned}$$

We conclude $g^L(N_\Phi(Y_1, Y_2), Z) = -d\eta(Y_1, Y_2)$ since $d(g^L(Z, \cdot))|_{\mathcal{L}^\perp} \in \Lambda^{1,1}_B \mathcal{X}|_{\mathcal{L}^\perp}$ and $d\eta = d(g^L(Z, \cdot))|_{\mathcal{L}^\perp}$. ☐

Prop. 2.76 raises the question whether X itself admits a complex or even a Kähler structure. In general, the answer to the latter question is negative.

In order to see this consider a compact simply connected Kähler manifold (M, g) such that $n := \dim_{\mathbb{C}} M$ and the compact toric type manifold $(X = \tilde{X} \times S^1, \tilde{g}_f)$ over (M, g) where $\pi : \tilde{X} \to M$ is the S^1-bundle corresponding to $0 \neq -[\frac{\psi}{2\pi}] \in H^2(M, \mathbb{Z})$. Suppose we have achieved $Hol(X, \tilde{g}_f) \subset U(n) \ltimes \mathbb{R}^n$ using one of the constructions from the last section. Using the conventions from that section $(X, \tilde{g}_f, V, S)$ is horizontal and $L_Z \tilde{J} = 0$. Moreover, Z is complete, but X does not admit a Kähler structure for the following reason:

If $\omega \in H^2(X, \mathbb{R})$ is a Kähler class then $0 \neq \omega^{n+1} \in H^{2n+2}(X, \mathbb{R})$. Gysin's sequence implies $0 \to H^1(\tilde{X}, \mathbb{R}) \to H^0(M, \mathbb{R}) = \mathbb{R} \xrightarrow{\neq 0} H^2(M, \mathbb{R})$ and $H^0(M, \mathbb{R}) \to H^2(M, \mathbb{R}) \xrightarrow{\pi^*} H^2(\tilde{X}, \mathbb{R}) \to H^1(M, \mathbb{R}) = 0$, i.e., $H^1(\tilde{X}, \mathbb{R}) = 0$ and $H^2(\tilde{X}, \mathbb{R}) \subset \pi^*(H^2(M, \mathbb{R}))$.

Hence, $\omega = \pi^*(\alpha)$ for some $\alpha \in H^2(M, \mathbb{R})$ and we have the contradiction $\omega^{n+1} = \pi^*(\alpha^{n+1}) = 0$ since $\alpha^{n+1} \in H^{\dim_{\mathbb{R}} M + 2}(M, \mathbb{R}) = 0$.

[22] In contrast to [BG08] we define $d\eta(Y_1, Y_2) := Y_1 \eta(Y_2) - Y_2 \eta(Y_1) - \eta([Y_1, Y_2])$.

2.4 Bochner's Technique for Decent Spacetimes

The goal of this section is to present Bochner's technique for decent spacetimes. The idea is to compute the cohomology of a decent spacetime (X, g^L, V) in three steps. The first step is to compute the cohomology of X in terms of that of a leaf $\mathcal{L}^\perp$ of $\mathcal{X}^\perp$ and the second step is to relate the basic cohomology of the Riemannian foliation $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ to the cohomology of $\mathcal{L}^\perp$. The final step is to achieve a curvature comparison result relating the curvature of ∇^L to basic cohomology.

However, Bochner's technique always needs some compactness assumptions as it involves a Hodge theorem. Moreover, we have already considered the case where all leaves of $\mathcal{X}|_{\mathcal{L}^\perp}$ are compact (cf. Thm. 1.30 and Prop. 2.59) and using mild completeness conditions we derived results for strongly causal spacetimes (cf. Thm. 2.66). Hence, we will focus on decent spacetimes for which all leaves of $\mathcal{X}^\perp$ are compact!

If (X, g^L, V) is a decent spacetime and S is any realization of the screen bundle then $(X, \mathcal{X}^\perp)$ is a transversely parallelizable Riemannian foliation by Lemma 2.47, i.e., $[T\mathcal{X}^\perp, Z] \subset T\mathcal{X}^\perp$ and all leaves of $\mathcal{X}^\perp$ have trivial leaf holonomy.[23] By Molino's theorem 1.30 (or [Sha97, Cor. 8.6]) we derive a smooth fiber bundle $X \to X/_{\mathcal{X}^\perp}$ where $X/_{\mathcal{X}^\perp} \in \{\mathbb{R}, S^1\}$. Since all leaves of $\mathcal{X}^\perp$ are compact we have $X/_{\mathcal{X}^\perp} = \mathbb{R}$ if X is non-compact. In this case, $X \cong \mathcal{L}^\perp \times \mathbb{R}$ and $b_i(X) = b_i(\mathcal{L}^\perp)$.

On the other hand, $X/_{\mathcal{X}^\perp} = S^1$ if X is compact. Hence, X is a mapping torus, i.e., if $\mathcal{L}^\perp$ is a leaf of $\mathcal{X}^\perp$ there is a diffeomorphism F of $\mathcal{L}^\perp$ such that $X = \mathcal{L}^\perp \times [0,1]/_\sim$ where $(p, 0) \sim (F(p), 1)$. Moreover, Cor. 2.50 implies $b_1(X) = b_1(\mathcal{L}^\perp) + 1$ and for the higher Betti numbers a Mayer-Vietoris argument (cf. [Hat02, Ex. 2.48]) yields the following exact sequence in real singular homology

$$\longrightarrow H_i(\mathcal{L}^\perp) \overset{Id - F_*^i}{\longrightarrow} H_i(\mathcal{L}^\perp) \overset{\iota_*}{\longrightarrow} H_i(X) \longrightarrow H_{i-1}(\mathcal{L}^\perp) \overset{Id - F_*^{i-1}}{\longrightarrow} H_{i-1}(\mathcal{L}^\perp) \longrightarrow$$

where F_*^i is the morphism induced by F and $\iota : \mathcal{L}^\perp \hookrightarrow X$ is the inclusion.

We continue with the second step.

Consider an arbitrary almost decent spacetime (X, g^L, V) and suppose $\mathcal{X}^\perp$ admits a compact leaf $\mathcal{L}^\perp$. By Thm. 1.34 there exists a bundle-like Riemannian metric g^{DM} on $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ whose mean curvature 1-form $\kappa_{g^{DM}}$ is basic and $\Delta_B^{g^{DM}}$-harmonic such that g^{DM} and $g^R|_{\mathcal{L}^\perp}$ induce the same transverse metric on the quotient bundle $\mathcal{S}|_{\mathcal{L}^\perp} = T\mathcal{L}^\perp/_{T\mathcal{X}|_{\mathcal{L}^\perp}}$.[24] In this case, the Euler form $\mathbf{e}$ of $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp}, g^{DM})$ is defined using Rummler's formula

$$d(g^{DM}(\frac{V}{\|V\|_{DM}}, \cdot)) = g^{DM}(\frac{V}{\|V\|_{DM}}, \cdot) \wedge \kappa_{g^{DM}} + \mathbf{e}.$$

Royo Prieto proved the existence of a Gysin sequence for Riemannian flows:

[23] This follows from the definition or using the linearized holonomy group in the same way as we did in Cor. 2.62 since Z is parallel w.r.t. the transverse Levi-Civita connection of $(X, \mathcal{X}^\perp)$.

[24] Notice that we did not claim g^{DM} to be induced by some realization of the screen bundle, i.e., in general $g^{DM} \neq g^R|_{\mathcal{L}^\perp}$.

Theorem 2.77 (Royo Prieto [RP01]). *If $(X, \mathcal{F})$ is a Riemannian flow on the compact manifold X with a bundle-like metric g whose mean curvature 1-form is basic and harmonic then there is the following exact sequence*

$$\cdots \longrightarrow H^i_B(X, \mathcal{F}) \longrightarrow H^i(X, \mathbb{R}) \longrightarrow H^{i-1}_{d-\kappa_g}(X, \mathcal{F}) \xrightarrow{\pm[\cdot \wedge \mathbf{e}]} H^{i+1}_B(X, \mathcal{F}) \longrightarrow \cdots$$

where $H^{i-1}_{d-\kappa_g}(X, \mathcal{F})$ is the dual basic cohomology of $(X, \mathcal{F}, g)$. □

In our setting Thm. 2.77 provides the long exact sequence

$$\longrightarrow H^i_B(\mathcal{X}|_{\mathcal{L}^\perp}) \longrightarrow H^i(\mathcal{L}^\perp, \mathbb{R}) \longrightarrow H^{i-1}_{d-\kappa_{g^{DM}}}(\mathcal{X}|_{\mathcal{L}^\perp}) \xrightarrow{\pm[\cdot \wedge \mathbf{e}]} H^{i+1}_B(\mathcal{X}|_{\mathcal{L}^\perp}) \longrightarrow$$

Remark 2.78. We have seen in Lemma 2.56 that $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp}, g^R|_{\mathcal{L}^\perp})$ is an isometric Riemannian flow if g^R is the (V, S)-metric of an almost horizontal spacetime (X, g^L, V, S). In this case, $\kappa_{g^R|_{\mathcal{L}^\perp}} = 0$ and the Gysin sequence (as well as its proof [BG08, Thm. 7.2.1]) simplifies to

$$\cdots \to H^i_B(\mathcal{X}|_{\mathcal{L}^\perp}) \to H^i(\mathcal{L}^\perp, \mathbb{R}) \to H^{i-1}_B(\mathcal{X}|_{\mathcal{L}^\perp}) \xrightarrow{\delta} H^{i+1}_B(\mathcal{X}|_{\mathcal{L}^\perp}) \to \cdots$$

where $\delta = [dg^L(Z, \cdot) \wedge \cdot]$. In particular, $[dg^L(Z, \cdot)|_{\mathcal{L}^\perp}] \in H^2_B(\mathcal{X}|_{\mathcal{L}^\perp})$ is the Euler class of $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp}, g^R|_{\mathcal{L}^\perp})$. □

Finally, we continue with the third step.

First, we need a technical observation. By definition, an almost decent spacetime (X, g^L, V) is time-orientable and since all Lorentzian manifolds in this section are supposed to be orientable manifolds we conclude $Hol(X, g^L) \subset SO_0(1, \dim X - 1)$. Using the matrix form of $Hol(X, g^L)$ we observe that $Hol(\nabla^S|_{\mathcal{L}^\perp}) \subset Hol(\nabla^S) \subset SO(\dim X - 2)$. By Prop. 2.54 we derive $Hol(\nabla^T) \subset SO(\dim \mathcal{L}^\perp - 1)$ for the transverse Levi-Civita connection of $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$. Hence, $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ is transversely orientable. Moreover, each leaf $\mathcal{L}^\perp$ itself is orientable since $Z \in \Gamma(X, TX)$ is a smooth vector field which is transverse to $\mathcal{X}^\perp$.

Definition 2.79. *Let $(X, \mathcal{F})$ be a transversely orientable Riemannian flow and g^{DM} as above.*

1. *The twisted basic cohomology $H^*_{tw}(X, \mathcal{F})$ is defined as the cohomology of the complex $(\mathcal{A}^k_\mathcal{F}(X), d_\kappa)$ where $d_\kappa := d - \frac{1}{2}\kappa_{g^{DM}} \wedge \cdot$.*

2. *If δ_κ is the formal L^2-adjoint of d_κ on $\mathcal{A}^k_\mathcal{F}(X)$ the twisted basic Laplacian is defined by $\Delta_\kappa := d_\kappa \delta_\kappa + \delta_\kappa d_\kappa$.*

Twisted basic cohomology was defined in [HR10] in order to introduce a basic cohomology theory satisfying Poincaré duality. In particular, Habib and Richardson proved a Hodge decomposition theorem for Δ_κ acting on $H^*_{tw}(X, \mathcal{F})$. Therefore, Thm. 1.36 remains true if we replace $\mathcal{H}^k(X, \mathcal{F}, g)$ by $\mathcal{H}^k_{tw}(X, \mathcal{F}, g^{DM}) := \{\alpha \in \mathcal{A}^k_\mathcal{F}(X) : \Delta_\kappa \alpha = 0\}$.

Proposition 2.80 ([HR10, Prop. 6.7] and [Jun01, Prop. 3.3])**.** *Let ∇^T be the transverse Levi-Civita connection of g^{DM} on basic forms and let ∇^{T^*} be the formal L^2-adjoint of ∇^T w.r.t. g^{DM}.*

1. $\Delta_\kappa\varphi = \nabla^{T^*}\nabla^T\varphi + \sum_{i,j} e^j \wedge e_i \lrcorner R^T(e_i,e_j)\varphi + \frac{1}{4}|\kappa_{g^{DM}}|^2\varphi \quad \forall\varphi \in \mathcal{A}_{\mathcal{F}}(X)$, *where* $R^T(e_i,e_j) := [\nabla^T_{e_i},\nabla^T_{e_j}] - \nabla^T_{[e_i,e_j]}$ *and* $(e_1,\dots,e_{\dim X-1})$ *is a transverse orthonormal frame.*[25]

2. $\langle\Delta_\kappa\varphi,\varphi\rangle = \langle\nabla^{T^*}\nabla^T\varphi,\varphi\rangle + Ric^T(\varphi^\sharp,\varphi^\sharp) + \frac{1}{4}|\kappa_{g^{DM}}|^2|\varphi|^2 \quad \forall\varphi \in \mathcal{A}^1_{\mathcal{F}}(X)$, *where* Ric^T *is the Ricci curvature of* ∇^T.

□

At this point we remind that neither Ric^T nor $\sum_{i,j} e^j \wedge e_i \lrcorner R^T(e_i,e_j)\varphi$ depend on the specific bundle-like metric g^{DM}, but only on the induced transverse metric on the quotient bundle $TX/T\mathcal{F}$ (cf. Sec. 1.3).

Let $(Y_1,\dots,Y_{\dim S})$ be a local orthonormal frame of a realization S of the screen bundle and write $E_\pm := \frac{1}{\sqrt{2}}(V \pm Z)$. We compute

$$\begin{aligned}
Ric^L(Y_\alpha,Y_\beta) &= -g^L(R^L(Y_\alpha,E_-)E_-,Y_\beta)) + \sum_{k=1}^{\dim S} g^L(R^L(Y_\alpha,Y_k)Y_k,Y_\beta) \\
&\qquad + g^L(R^L(Y_\alpha,E_+)E_+,Y_\beta)) \\
&= \underbrace{g^L(R^L(Y_\alpha,V)Z,Y_\beta)}_{=-g^L(R^L(Z,Y_\beta)V,Y_\alpha)} + g^L(\underbrace{R^L(Y_\alpha,Z)V}_{\in\Xi},Y_\beta) \\
&\qquad + \sum_{k=1}^{\dim S} g^L(R^L(Y_\alpha,Y_k)Y_k,Y_\beta) \\
&= \sum_{k=1}^{\dim S} g^R(R^S(Y_\alpha,Y_k)Y_k,Y_\beta) = Ric^T(Y_\alpha,Y_\beta)
\end{aligned}$$

for the Ricci curvature Ric^L of (X,g^L) where we applied Prop. 2.54 for the last equation. Notice that $Ric^L(V,\cdot)|_{\Xi^\perp} = 0$ for any almost decent spacetime.

Proposition 2.81. *Let (X,g^L,V) be an almost decent spacetime and $\mathcal{L}^\perp$ a compact leaf of $\mathcal{X}^\perp$. If $Ric^L(W,W) \geq 0$ for all $W \in T\mathcal{L}^\perp$ then $b_1(\mathcal{L}^\perp) \leq \dim\mathcal{L}^\perp$. If additionally $Ric^L_q(W,W) > 0$ for some $q \in \mathcal{L}^\perp$ and all $W \in S_q$ then $b_1(\mathcal{L}^\perp) \leq 1$.*

Proof. Suppose $Ric^L(W,W) \geq 0$ and let g^{DM} be a bundle-like Riemannian metric on $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ having a basic and harmonic mean curvature form $\kappa_{g^{DM}}$. By the transverse Hodge theorem 1.36 a class $[\varphi] \in H^1_B(\mathcal{L}^\perp,\mathcal{X}|_{\mathcal{L}^\perp})$ can be represented by a basic 1-form φ such that $d\varphi = \delta_B\varphi = 0$.

[25]Note that $\nabla^{T^*}\nabla^T := \sum_i \nabla^{T^*}_{e_i}\nabla^T_{e_i} := -\sum_i \nabla^T_{e_i}\nabla^T_{e_i} - \nabla^T_{\nabla^T_{e_i}e_i} - \nabla^T_{\kappa^\sharp}$.

In this case, $d_\kappa\varphi = -\frac{1}{2}\kappa_{g^{DM}} \wedge \varphi$ and $\delta_\kappa\varphi = -\frac{1}{2}\kappa_{g^{DM}} \lrcorner\varphi$ where $\kappa_{g^{DM}} \lrcorner\cdot$ is the pointwise adjoint of $\kappa_{g^{DM}} \wedge \cdot$. Therefore, the Weitzenböck formula has the form $0 = \int_{\mathcal{L}^\perp} |\nabla^T\varphi|^2 + \int_{\mathcal{L}^\perp} Ric^T(\varphi^\sharp, \varphi^\sharp)$ (cf. [HR10, Thm. 6.16]). Hence, $\nabla^T_{e_i}\varphi = 0$ for all $1 \leq i \leq \dim \mathcal{L}^\perp - 1$. However, $\nabla^T_V\varphi = 0$ since φ is basic and we have $\dim H^1_B(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp}) \leq \dim \mathcal{S}$ for dimensional reasons.

If $Ric^L_q(Y,Y) > 0$ at $q \in \mathcal{L}^\perp$ the above equation implies $H^1_B(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp}) = 0$. Since $H^0_{d-\kappa_{g^{DM}}}(\mathcal{X}|_{\mathcal{L}^\perp}) = H_B^{\dim \mathcal{L}^\perp - 1}(\mathcal{X}|_{\mathcal{L}^\perp}) \in \{\mathbb{R}, 0\}$ by Prop. 1.37 and

$$0 \to H^1_B(\mathcal{X}|_{\mathcal{L}^\perp}) \to H^1(\mathcal{L}^\perp, \mathbb{R}) \to H^0_{d-\kappa_{g^{DM}}}(\mathcal{X}|_{\mathcal{L}^\perp}) \overset{[\cdot\wedge\mathbf{e}]}{\to} H^2_B(\mathcal{X}|_{\mathcal{L}^\perp})$$

we conclude $b_1(\mathcal{L}^\perp) \leq \dim H^1_B(\mathcal{X}|_{\mathcal{L}^\perp}) + 1$. □

Putting all three steps together we derive the following Bochner technique for decent spacetimes.

Theorem 2.82. *Let (X, g^L, V) be a decent spacetime and $\mathcal{L}^\perp$ a leaf of $\mathcal{X}^\perp$. Suppose $Ric^L(W,W) \geq 0$ for all $W \in T\mathcal{L}^\perp$.*

1. *If X is compact and $\mathcal{X}^\perp$ admits a compact leaf then $1 \leq b_1(X) \leq \dim X$.*
2. *If X is non-compact and all leaves of $\mathcal{X}^\perp$ are compact then $0 \leq b_1(X) \leq \dim X - 1$.*

Moreover, if $Ric^L_q(W,W) > 0$ for some $q \in \mathcal{L}^\perp$ and all $W \in S_q$ the bounds are $1 \leq b_1(X) \leq 2$ and $0 \leq b_1(X) \leq 1$ respectively.

Proof. Using the Mayer-Vietoris argument and Prop. 2.81 we derive $b_1(X) \leq b_1(\mathcal{L}^\perp) + 1 \leq \dim X$ if X is compact. In the non-compact case we observed $X \cong \mathcal{L}^\perp \times \mathbb{R}$, i.e., $b_1(X) = b_1(\mathcal{L}^\perp) \leq \dim X - 1$. □

Conversely, we may apply toric type manifolds to prove

Proposition 2.83. *The bounds in Theorem 2.82 are optimal.*

Proof. First, we consider the upper bounds.

1. If (M,g) is a compact Riemannian manifold we consider $X = S^1 \times M \times S^1$ resp. $X = S^1 \times M \times \mathbb{R}$ with the following Lorentzian metrics: If ∂_x is the coordinate field on S^1 define $g^L := 2dxdz + fdz^2 + g$ where ∂_z is the coordinate field of the last factor and $f \in C^\infty(M)$. Thus, $\Xi = TS^1$, $\mathcal{L}^\perp \cong S^1 \times M$ and $\nabla^S|_{\mathcal{L}^\perp}$ is flat if (M,g) is the flat torus. In particular, $b_1(S^1 \times M \times S^1) = \dim M + 2$ and $b_1(S^1 \times M \times \mathbb{R}) = \dim M + 1$.
2. For the second statement let (M,g) be a compact simply connected Riemannian manifold with strictly positive Ricci curvature and let (X, g^L) be as above. Then $Ric^T > 0$ and the bounds follow.

Finally, we consider the lower bounds.

1. Let (M, g) be a compact simply connected Calabi-Yau manifold, i.e., $Hol(M, g) = SU(n)$ and suppose $b_2(M) \neq 0$. Consider the toric type Lorentzian manifold $(X, \tilde{g}_f)$ which is given as in Prop. 2.30 with $0 \neq c_1(\tilde{X} \to M) \in H^{1,1}_{prim}(M) \cap H^2(M, \mathbb{Z})$. Thus, $Hol(X, g^L) = SU(n) \times \mathbb{R}^{2n}$ and $\mathcal{L}^\perp \cong \tilde{X}$. In particular, $\nabla^S|_{\mathcal{L}^\perp}$ is Ricci flat and the Gysin sequence for the S^1-bundle $\tilde{X} \to M$ implies

$$0 = H^1(M, \mathbb{R}) \longrightarrow H^1(\tilde{X}, \mathbb{R}) \longrightarrow \mathbb{R} = H^0(M, \mathbb{R}) \xrightarrow{c_1} H^2(M, \mathbb{R}),$$

i.e., $b_1(\tilde{X}) = 0$ and therefore $b_1(X) = 1$ if X is compact and $b_1(X) = 0$ otherwise.

2. Next, let (M, g) be a compact simply connected Riemannian manifold with strictly positive Ricci curvature and let $\alpha \in H^2(M, \mathbb{Z})$ be a generator. If $(X, \tilde{g}_f)$ is of toric type over (M, g) where $c_1(\tilde{X} \to M) = \alpha$ then $Ric^T|_{S \times S} = Ric(M, g) > 0$ by the discussion preceding Prop. 2.25. Since $\mathcal{L}^\perp \cong \tilde{X}$ and $b_1(M) = 0$ Gysin's sequence implies $b_1(\tilde{X}) = 0$ and the bounds follow.

□

In some special cases the condition $Ric^L(W, W) \geq 0$ for all $W \in T\mathcal{L}^\perp$ has a natural meaning.

Definition 2.84. *Let (X, g^L) be a Lorentzian manifold. We say (X, g^L) satisfies the strong energy resp. timelike convergence condition at $p \in X$ if $Ric^L_p(W, W) \geq 0$ for any timelike vector $W \in T_pX$.*

If (X, g^L, V) is a decent spacetime such that $\nabla^L_\cdot V = 0$ then a short computation shows

$$\begin{aligned} Ric^L(V, \cdot) &= 0, \\ Ric^L(Z, Z) &= \sum_{k=1}^{\dim X - 2} g^L(R^L(Z, Y_k)Y_k, Z), \\ Ric^L(Z, Y_i) &= \sum_{k=1}^{\dim X - 2} g^L(R^S(Z, Y_k)Y_k, Y_i). \end{aligned}$$

We have already proved $Ric^L(Y_\alpha, Y_\beta) = \sum_{k=1}^{\dim X - 2} g^S(R^L(Y_\alpha, Y_k)Y_k, Y_\beta)$. Hence, we conclude

Remark 2.85. Let (X, g^L, V) be a decent spacetime such that $\nabla^L_\cdot V = 0$ and let $p \in X$. If $Ric^L_p(Z, Z) = 0$ and $\sum_k R^S_p(Z, Y_k)Y_k = 0$ then (X, g^L) satisfies the strong energy condition at p if and only if $Ric^L_p(W, W) \geq 0$ for all $W \in \Xi^\perp_p$. □

Note that the spacetimes considered in the first part of the proof of Prop. 2.83 provide examples for the preceding remark.

For the rest of this section we study restriction on the compact foliated manifold $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ which are induced by the screen holonomy of (X, g^L, V).

Corollary 2.86. *Let (X, g^L, V) be an almost decent spacetime and let $\mathcal{L}^\perp$ be a compact leaf of $\mathcal{X}^\perp$. If $Ric^L(W, W) \geq 0$ for all $W \in T\mathcal{L}^\perp$ and $Hol(\nabla^S|_{\mathcal{L}^\perp})$ acts irreducibly then $H^1_B(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp}) = 0$ and $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ is taut.*

Proof. Let $[\varphi] \in H^1_B(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ be represented by a basic $\Delta_B^{g^{DM}}$-harmonic 1-form φ. As in Prop. 2.81 the Weitzenböck formula has the form $0 = \int_{\mathcal{L}^\perp} |\nabla^T\varphi|^2 + \int_{\mathcal{L}^\perp} Ric^T(\varphi^\sharp, \varphi^\sharp)$. Since φ cannot be $\nabla^S|_{\mathcal{L}^\perp}$-parallel we have $\varphi = 0$. In particular, the Álvarez-class $[\kappa_{g^{DM}}] \in H^1_B(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ vanishes and Prop. 1.37 implies the statement. $\square$

The next result applies the transverse Kähler identities which are proved in [EKA90] and have a similar form as those in Prop. 1.19.

Lemma 2.87. *Let (X, g^L, V) be an almost decent spacetime and $\mathcal{L}^\perp$ a compact leaf of $\mathcal{X}^\perp$. If $Ric^L|_{T\mathcal{L}^\perp \times T\mathcal{L}^\perp} = 0$ and $Hol(\nabla^S|_{\mathcal{L}^\perp}) \subset U(n)$ then any basic $(p,0)$-form ψ on $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ is closed if and only if $\nabla^S|_{\mathcal{L}^\perp}\psi = 0$.*

Proof. One part of the proof is implied by $d\psi = \sum_{i=1}^{\dim S} e^i \wedge \nabla^S_{e_i}\psi$.

Since ψ is a $(p,0)$-form we have $\bar{\delta}_b\psi = 0$ and $d\psi = 0$ implies $\bar{\partial}\psi = 0$, i.e., $\Delta_b^{\bar{\partial}}\psi = 0$. Thus, $\Delta_b\psi = 0$ by the transverse Kähler identities and we have $d\psi = \delta_b\psi = 0$ implying $\int_{\mathcal{L}^\perp} \langle \Delta_\kappa \psi, \psi\rangle = \frac{1}{4}|\kappa|^2|\psi|^2$. By the Weitzenböck formula

$$0 = \int_{\mathcal{L}^\perp} |\nabla^T\psi|^2 + \int_{\mathcal{L}^\perp} \langle \textstyle\sum_{i,j} e^j \wedge e_i \lrcorner R^T(e_i, e_j)\psi, \psi\rangle.$$

However, R^T being the curvature of $\nabla^S|_{\mathcal{L}^\perp}$ has the same symmetries as the curvature tensor of a Kähler manifold. Using the computation in [Joy00, Prop. 6.2.4] we conclude $\sum_{i,j} e^j \wedge e_i \lrcorner R^T(e_i, e_j)\psi = 0$, i.e., $0 = \int_{\mathcal{L}^\perp} |\nabla^T\psi|^2$. $\square$

Proposition 2.88. *Let (X, g^L, V) be a decent spacetime and $\mathcal{L}^\perp$ a compact leaf of $\mathcal{X}^\perp$. Suppose $Hol(\nabla^S|_{\mathcal{L}^\perp})$ is irreducible and $Ric^L(W, W) \geq 0$ for all $W \in T\mathcal{L}^\perp$.*

1. *If X is compact then $b_1(X) \in \{1, 2\}$ and $b_2(X) \leq \dim H^2_B(\mathcal{X}|_{\mathcal{L}^\perp}) + 1$.*
2. *If X is non-compact and all leaves of $\mathcal{X}^\perp$ are compact then $b_1(X) \in \{0, 1\}$ and $b_2(X) \in \{\dim H^2_B(\mathcal{X}|_{\mathcal{L}^\perp}) - 1, \dim H^2_B(\mathcal{X}|_{\mathcal{L}^\perp})\}$.*

Moreover, if $Hol(\nabla^S|_{\mathcal{L}^\perp}) = SU(n)$ with $n \geq 3$ we can replace $H^2_B(\mathcal{X}|_{\mathcal{L}^\perp})$ by $H^{1,1}_B(\mathcal{X}|_{\mathcal{L}^\perp})$ and in the case that $Hol(\nabla^S|_{\mathcal{L}^\perp}) = Sp(n)$ with $n \geq 1$ we can replace $\dim H^2_B(\mathcal{X}|_{\mathcal{L}^\perp})$ by $\dim H^{1,1}_B(\mathcal{X}|_{\mathcal{L}^\perp}) + 2$.

Proof. Since $\dim H^1_B(\mathcal{X}|_{\mathcal{L}^\perp}) = 0$ and $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ is taut by Cor. 2.86 we derive the bounds for $b_1(X)$ and the Gysin sequence implies

$$\mathbb{R} \xrightarrow{[\cdot\wedge\mathbf{e}]} H^2_B(\mathcal{X}|_{\mathcal{L}^\perp}) \longrightarrow H^2(\mathcal{L}^\perp, \mathbb{R}) \longrightarrow 0,$$

i.e., $b_2(\mathcal{L}^\perp) \in \{\dim H^2_B(\mathcal{X}|_{\mathcal{L}^\perp}) - 1, \dim H^2_B(\mathcal{X}|_{\mathcal{L}^\perp})\}$. If X is compact the Mayer-Vietoris argument implies

$$H_2(\mathcal{L}^\perp) \xrightarrow{Id - F^2_*} H_2(\mathcal{L}^\perp) \xrightarrow{\iota_*} H_2(X) \longrightarrow H_1(\mathcal{L}^\perp) \xrightarrow{Id - F^1_*} H_1(\mathcal{L}^\perp).$$

Hence, $b_2(X) = b_1(\mathcal{L}^\perp) + \dim \mathrm{Eig}_1(F^2_*)$, where $\mathrm{Eig}_1(F^2_*)$ is the eigenspace of F^2_* w.r.t. the eigenvalue 1, i.e., $b_2(X) \leq b_1(\mathcal{L}^\perp) + b_2(\mathcal{L}^\perp) \leq \dim H^2_B(\mathcal{X}|_{\mathcal{L}^\perp}) + 1$. The remaining statements follow from Lemma 2.87. $\square$

3 Submanifolds in Spaces of Constant Curvature

3.1 The Normal Holonomy of Good Submanifolds

In this chapter we study screen trees associated to the normal bundle of submanifolds in pseudo-Riemannian spaces of constant curvature. The purpose of the first section is to define (very) good submanifolds and to classify the leaves of its screen trees. We start to introduce our notation. For more comprehensive introductions to submanifolds we refer to [BCO03] and [Bes87].

Definition 3.1. *Let $f : (X, g) \to (Y, h)$ be a smooth map between pseudo-Riemannian manifolds. Then we say f is a pseudo-Riemannian*[1]

- *submanifold if $f^*h = g$,*
- *immersed submanifold if $f^*h = g$ and f is injective,*
- *embedded submanifold if $f^*h = g$ and $f : X \to f(X) \subset Y$ is a homeomorphism.*

In the following we assume $0 < \dim_{\mathbb{R}} X < \dim_{\mathbb{R}} Y$. By abuse of notation we will say (X, g) is a (non-degenerate pseudo-Riemannian) submanifold of (Y, h) since we do not consider degenerate submanifolds. All definitions are inequivalent and $f(X) \subset Y$ is not necessarily a manifold if X is a submanifold of Y. However, locally a submanifold is an embedded submanifold.

We write $TY|_X := f^*TY$ and using f_* we consider TX as a subbundle in $TY|_X$. The normal bundle NX of $f : (X, g) \to (Y, h)$ is the h-orthogonal complement of $TX \subset TY|_X$. Hence, we have an orthogonal decomposition

$$TY|_X = TX \oplus NX.$$

If ∇^X and ∇^Y are the Levi-Civita connections on (X, g) and (Y, h) then $pr_{TX} \circ \nabla^Y|_{TX} = \nabla^X$. The projections define the second fundamental form $\mathrm{II} : TX \times TX \to NX$ of f by

$$\nabla^Y_V W = \nabla^X_V W + \mathrm{II}(V, W), \quad V, W \in \Gamma(TX)$$

and the induced connection $\nabla^\perp$ on the normal bundle NX by

$$\nabla^Y_V \xi = -A_\xi V + \nabla^\perp_V \xi, \quad V \in \Gamma(TX),\ \xi \in \Gamma(NX),$$

[1] It is certainly unusual to define a submanifold as a morphism, but it simplifies the presentation in this chapter.

where $A_\xi V := -pr_{TX}(\nabla^Y_\xi V)$ is the shape operator of f. Since $\nabla^\perp$ is a metric connection w.r.t. the induced metric on NX we derive a pseudo-Riemannian vector bundle $(NX, h|_{NX}, \nabla^\perp, \pi, X)$. Finally, $h(\Pi(V,W),\xi) = g(A_\xi V, W)$ and we have $g(A_\xi V, W) = g(V, A_\xi W)$, i.e., A_ξ is g-self-adjoint on TX.

Definition 3.2. *Let $f : (X, g) \to (Y, h)$ be a submanifold.*

1. *The codimension of X in Y is the rank of the normal bundle NX.*

2. *The cosignature of X in Y is the signature of $(NX, h|_{NX})$.*

3. *We say f is a simply connected submanifold if $\pi_1(X) = 0$.*

4. *The normal holonomy group at $p \in X$ is given by $Hol_p(\nabla^\perp)$. The same way we define the (local) normal holonomy algebra and its representations.*

5. *The normal curvature tensor $R^\perp_p : \Lambda^2 T_pX \times N_pX \to N_pX$ at $p \in X$ is given by $R^\perp(V,W) := [\nabla^\perp_V, \nabla^\perp_W] - \nabla^\perp_{[V,W]}$.*

At this point remind that the normal holonomy group is invariant under conformal diffeomorphisms of the ambient space.[2] Therefore, all classification results for the normal holonomy representation of submanifolds in spaces of constant curvature carry over to the local normal holonomy representation of submanifolds in locally conformally flat pseudo-Riemannian spaces.

If the submanifold $f : (X, g) \to (Y, h)$ is not simply connected and $F : \tilde{X} \to X$ is the universal covering of X then $f \circ F : (\tilde{X}, \tilde{g}) \to (Y, h)$ is a submanifold. Furthermore, $(F^*NX, F^*h|_{NX}, F^*\nabla^{\perp,f}, F^*\pi, \tilde{X})$ is naturally isomorphic to $(N\tilde{X}, h|_{N\tilde{X}}, \nabla^{\perp, f\circ F}, \tilde{\pi}, \tilde{X})$ since F is a local diffeomorphism. Thus, we can identify the full holonomy representation $Hol(\nabla^{\perp, f\circ F})$ with the restricted holonomy representation $Hol^0(\nabla^{\perp,f})$.

For any pseudo-Riemannian submanifold the normal curvature tensor and the shape operators are related by the *Ricci equation*

$$h(R^\perp(V,W)\xi_1, \xi_2) = g([A_{\xi_1}, A_{\xi_2}]V, W) + h(R^{\nabla^Y}(V,W)\xi_1, \xi_2).$$

In order to apply the theory of Berger algebras to $(NX, h|_{NX}, \nabla^\perp, \pi, X)$ we need to construct algebraic curvature tensors on NX with values in the normal holonomy algebra. We will consider the case where $(Y, h = \langle\cdot,\cdot\rangle)$ is a space of constant curvature, i.e.,

[2] This is well known (cf. [DS00]), but I could not find a reference including a proof. Hence, we prove this simple fact here. Let $\tilde{h} := e^{2f}h$ for $f \in C^\infty(Y)$ and $\gamma : I \to X$ be a closed curve in X. For the Levi-Civita connections of h and $\tilde{h}$ on Y we have [Bes87][1.159]

$$\nabla^{\tilde{h}}_V \xi = \nabla^h_V \xi + df(V)\xi + df(\xi)V - h(V,\xi)\mathrm{grad}_h(f).$$

If ξ, η are normal vector fields along γ and $V = \dot{\gamma}$ we have $\tilde{h}(\nabla^{\tilde{h}}_{\dot{\gamma}}\xi, \eta) = e^{2f}h(\nabla^h_{\dot{\gamma}}\xi, \eta) + df(\dot{\gamma})\tilde{h}(\xi,\eta)$. If ξ is $\nabla^{\perp,h}$-parallel along γ we conclude that $e^{-f}\xi$ is $\nabla^{\perp,\tilde{h}}$-parallel along γ. Hence, the parallel displacement of $\xi_0 \in N_{\gamma(0)}X$ w.r.t. $\nabla^{\perp,h}$ and $\nabla^{\perp,\tilde{h}}$ coincide along a closed curve.

$\langle R^{\nabla^Y}(V,W)\xi_1,\xi_2\rangle = 0$. However, in some cases this restriction can be dropped with minor modification of the ideas for the proofs (cf. [ADS04]). For $p \in X$ let $(e_1,\dots,e_{\dim X})$ be a pseudo-orthonormal basis of T_pX and $\xi_1,\xi_2,\xi_3 \in N_pX$. Define

$$\boxed{\mathcal{R}_p(\xi_1,\xi_2)\xi_3 := \sum_{i=1}^{\dim X} \varepsilon_i R_p^{\perp}(A_{\xi_1}e_i, A_{\xi_2}e_i)\xi_3.}$$

We refer to $\mathcal{R}_p$ as the *normal algebraic curvature tensor* at $p \in X$. It has been defined by Olmos in [Olm90] in order to classify restricted normal holonomy representations in the Riemannian context. Applying his ideas we can prove

Lemma 3.3. *Let $f : (X,g) \to (Y,\langle\cdot,\cdot\rangle)$ be a submanifold in a space of constant curvature $(Y,\langle\cdot,\cdot\rangle)$ and let $\mathcal{R}_p$ be its normal algebraic curvature tensor.*

1. *For $\xi_1,\dots,\xi_4 \in N_pX$ we have*
$$\langle \mathcal{R}_p(\xi_1,\xi_2)\xi_3,\xi_4\rangle = \frac{1}{2}Tr([A_{\xi_1},A_{\xi_2}]\circ[A_{\xi_3},A_{\xi_4}]).$$
2. *$\mathcal{R}_p$ is an algebraic curvature tensor on (N_pX,h) with values in $\mathfrak{so}(N_pX,h)$ in the sense of Definition 1.4.*
3. *For the generated endomorphisms the following holds*
$$span\{\mathcal{R}_p(\xi_1,\xi_2) : \xi_1,\xi_2 \in N_pX\} \subset span\{R_p^{\perp}(V,W) : V,W \in T_pX\}.$$

Proof. As we have mentioned the proof is similar to [Olm90]. Using the Ricci equation and the self-adjointness of the shape operators we have

$$\begin{aligned}
\langle \mathcal{R}_p(\xi_1,\xi_2)\xi_3,\xi_4\rangle &= \sum_{i=1}^{\dim X}\varepsilon_i\langle R_p^{\perp}(A_{\xi_1}e_i,A_{\xi_2}e_i)\xi_3,\xi_4\rangle\\
&= \sum_{i=1}^{\dim X}\varepsilon_i\langle [A_{\xi_3},A_{\xi_4}]A_{\xi_1}e_i, A_{\xi_2}e_i\rangle\\
&= \frac{1}{2}\sum_{i=1}^{\dim X}\varepsilon_i\langle A_{\xi_2}[A_{\xi_3},A_{\xi_4}]A_{\xi_1}e_i, e_i\rangle\\
&\quad - \frac{1}{2}\sum_{i=1}^{\dim X}\varepsilon_i\langle e_i, A_{\xi_1}[A_{\xi_3},A_{\xi_4}]A_{\xi_2}e_i\rangle\\
&= \frac{1}{2}Tr(A_{\xi_2}\circ[A_{\xi_3},A_{\xi_4}]\circ A_{\xi_1}) - \frac{1}{2}Tr(A_{\xi_1}\circ[A_{\xi_3},A_{\xi_4}]\circ A_{\xi_2})\\
&= \frac{1}{2}Tr(A_{\xi_1}\circ A_{\xi_2}\circ[A_{\xi_3},A_{\xi_4}]) - \frac{1}{2}Tr(A_{\xi_2}\circ A_{\xi_1}\circ[A_{\xi_3},A_{\xi_4}])\\
&= \frac{1}{2}Tr([A_{\xi_1},A_{\xi_2}]\circ[A_{\xi_3},A_{\xi_4}]).
\end{aligned}$$

Using this equation the Bianchi identity for $\mathcal{R}_p$ follows by straightforward computation since $Tr(A \circ B) = Tr(B \circ A)$. For the last statement we remind the notation from Appendix 3.2. More precisely, we have the isomorphism $F_p : \Lambda^2 T_p X \to \mathfrak{so}(T_p X, g)$ and define

$$G_p : \Lambda^2 N_p X \to \mathfrak{so}(T_p X, g), \qquad \xi \wedge \eta \mapsto [A_\xi, A_\eta].$$

Then, the Lemma follows from $\mathcal{R}_p = -R_p^\perp \circ F_p^{-1} \circ G_p$. In order to derive this identity we use $A_\xi(e_j) = \sum_{k=1}^{\dim X} \varepsilon_k \langle A_\xi e_j, e_k \rangle e_k$ and compute

$$\begin{aligned}
(F_p^{-1} \circ G_p)(\xi \wedge \eta) &= \sum_{i<j} \varepsilon_i \varepsilon_j \langle [A_\xi, A_\eta] e_i, e_j \rangle e_i \wedge e_j \\
&= \sum_{i<j} \varepsilon_i \varepsilon_j \langle A_\xi A_\eta e_i, e_j \rangle e_i \wedge e_j - \sum_{i<j} \varepsilon_i \varepsilon_j \langle A_\eta A_\xi e_i, e_j \rangle e_i \wedge e_j \\
&= \sum_{i<j} \varepsilon_i \varepsilon_j \langle A_\eta e_i, A_\xi e_j \rangle e_i \wedge e_j - \sum_{i<j} \varepsilon_i \varepsilon_j \langle A_\xi e_i, A_\eta e_j \rangle e_i \wedge e_j \\
&= \sum_{i \neq j} \varepsilon_i \varepsilon_j \langle A_\eta e_i, A_\xi e_j \rangle e_i \wedge e_j \\
&= \sum_{k=1}^{\dim X} \sum_{i \neq j} \varepsilon_k \varepsilon_i \varepsilon_j \langle A_\eta e_i, e_k \rangle \langle e_k, A_\xi e_j \rangle e_i \wedge e_j \\
&= \sum_{k=1}^{\dim X} \sum_{i \neq j} \varepsilon_k \varepsilon_i \varepsilon_j \langle A_\eta e_k, e_i \rangle \langle e_j, A_\xi e_k \rangle e_i \wedge e_j \\
&= \sum_{k=1}^{\dim X} \varepsilon_k A_\eta(e_k) \wedge A_\xi(e_k).
\end{aligned}$$

Hence, $(R_p^\perp \circ F_p^{-1} \circ G_p)(\xi, \eta) = \sum_{k=1}^{\dim X} \varepsilon_k R_p^\perp(A_\eta(e_k), A_\xi(e_k))$. □

Since $\mathrm{span}\{\mathcal{R}_p(\cdot,\cdot)\} \subset \mathrm{span}\{R_p^\perp(\cdot,\cdot)\}$ the Ambrose-Singer theorem implies

$$\mathrm{span}\{\mathcal{R}_p^{\tau_\gamma^\perp}(\tau_\gamma^\perp \xi, \tau_\gamma^\perp \eta) : \xi, \eta \in N_p X, \gamma : [0,1] \to X,\ \gamma(0) = p\} \subset \mathfrak{hol}_p(\nabla^\perp),$$

where $\mathcal{R}_p^{\tau_\gamma^\perp} = {\tau_\gamma^\perp}^{-1} \circ \mathcal{R}_{\gamma(1)} \circ \tau_\gamma^\perp$ and $\tau_\gamma^\perp$ is the parallel displacement along γ w.r.t. $\nabla^\perp$.

Definition 3.4. *Let $f : (X, g) \to (Y, \langle\cdot,\cdot\rangle)$ be a submanifold of signature (r, s) in a space of constant curvature $(Y, \langle\cdot,\cdot\rangle)$ and let $R_p^\perp$ be its normal curvature tensor. Define $K_p := Ker(R_p^\perp \circ F^{-1})$. Then, we say X is a*

- *good submanifold if for all $p \in X$ the subspace $K_p \subset \mathfrak{so}(T_p X, g)$ is non-degenerate w.r.t. the Killing form on $\mathfrak{so}(T_p X, g)$.*
- *very good submanifold if for all $p \in X$ the subspace $K_p^\perp \subset \mathfrak{so}(T_p X, g)$ is definite w.r.t. the Killing form on $\mathfrak{so}(T_p X, g)$.*

Lemma 3.5. *Let $f : (X, g) \to (Y, \langle\cdot,\cdot\rangle)$ be a submanifold in a space of constant curvature $(Y, \langle\cdot,\cdot\rangle)$. Then (X, g) is a good submanifold if and only if*

$$span\{\mathcal{R}_p(\xi_1, \xi_2) : \xi_1, \xi_2 \in N_pX\} = span\{R_p^{\perp}(V, W) : V, W \in T_pX\} \quad \forall p \in X.$$

In this case, $(NX, h|_{NX}, \nabla^{\perp}, \pi, X)$ has good holonomy.

Proof. Let $w := \sum_{i<j} a_{ij} e_i \wedge e_j$. Using the notation from Appendix 3.2 as well as

$$\langle F(e_i \wedge e_j)e_k, e_\ell\rangle = \varepsilon_k \varepsilon_\ell (\delta_{ik}\delta_{j\ell} - \delta_{jk}\delta_{i\ell})$$

and the Ricci equation we compute

$$\begin{aligned}
\langle R_p^{\perp}(w)\xi, \eta\rangle &= \sum_{i<j} a_{ij} \langle R_p^{\perp}(e_i, e_j)\xi, \eta\rangle = \sum_{i<j} a_{ij} \langle [A_\xi, A_\eta](e_i), e_j\rangle \\
&= \frac{1}{2} \sum_{k,\ell} \varepsilon_k \varepsilon_\ell \langle F_p(w)e_k, e_\ell\rangle \langle [A_\xi, A_\eta]e_k, e_\ell\rangle \\
&= -\frac{1}{2} \sum_{k,\ell} \varepsilon_\ell \langle F_p(w)e_k, e_\ell\rangle \varepsilon_k \langle [A_\xi, A_\eta]e_\ell, e_k\rangle \\
&= -\frac{1}{2} Tr(F_p(w) \circ [A_\xi, A_\eta]) \\
&= -\frac{1}{2} Tr(F_p(w) \circ G_p(\xi \wedge \eta)) \\
&= -B(w, (F_p^{-1} \circ G_p)(\xi \wedge \eta)).
\end{aligned}$$

Hence, $(\mathrm{Ker}(R_p^{\perp}))^{\perp_B} = \mathrm{Im}(F_p^{-1} \circ G_p)$. As we have seen in Appendix 3.2 $\mathrm{Ker}(R_p^{\perp})$ is non-degenerate w.r.t. B if and only if $\mathrm{Ker}(R_p^{\perp} \circ F^{-1})$ is non-degenerate w.r.t. the Killing form on $\mathfrak{so}(T_pX, g)$. If this is the case we conclude $R_p^{\perp}(w) \in \mathrm{Im}(R_p^{\perp} \circ F_p^{-1} \circ G_p)$. On the other hand if $\mathrm{Ker}(R_p^{\perp})$ is degenerate we still have $\dim \mathrm{Im}(F_p^{-1} \circ G_p) = \dim \Lambda^2 T_pX - \dim \mathrm{Ker}(R_p^{\perp}) = \dim \mathrm{Im}(R_p^{\perp})$ by orthogonality. Thus, $\dim \mathrm{Coker}(\mathrm{Im}(F_p^{-1} \circ G_p) \cap \mathrm{Ker}(R_p^{\perp}) \hookrightarrow \mathrm{Im}(F_p^{-1} \circ G_p)) < \dim \mathrm{Im}(R_p^{\perp})$ since $\mathrm{Im}(F_p^{-1} \circ G_p) \cap \mathrm{Ker}(R_p^{\perp}) = (\mathrm{Ker}(R_p^{\perp}))^{\perp_B} \cap \mathrm{Ker}(R_p^{\perp}) \neq \{0\}$. This implies the converse.

Finally, we apply the Ambrose-Singer theorem to conclude that $(NX, h|_{NX}, \nabla^{\perp}, \pi, X)$ has good holonomy. □

A submanifold $f : (X, g) \to (Y, h)$ in a pseudo-Riemannian manifold is said to be spacelike if (X, g) is a Riemannian manifold. The Killing form on $\mathfrak{so}(T_pX, g)$ of a spacelike submanifold is definite implying

Corollary 3.6. *Any spacelike submanifold $f : (X, g) \to (Y, \langle\cdot,\cdot\rangle)$ in a space of constant curvature is a very good submanifold.* □

Remark 3.7. It is known that with a possibly different signature any pseudo-Riemannian manifold (X, g) can be isometrically embedded into some $\mathbb{R}^{r,s}$. For this result we refer to [MS08b] and the references therein. So far, we do not know whether the embedding can be chosen to be good. However, we may easily derive examples since by a short computation and Corollary 3.6 the extrinsic product $(f_1, f_2) : X_1 \times X_2 \to \mathbb{R}^{r,s}$ is a very good pseudo-Riemannian submanifold if $f_1 : X_1 \to \mathbb{R}^{r,0}$ and $f_2 : X_2 \to \mathbb{R}^{0,s}$ are submanifolds. $\square$

Corollary 3.8. *Let $f : (X, g) \to (Y, \langle\cdot,\cdot\rangle)$ be a good submanifold in a space of constant curvature and $p \in X$.*

1. *Then $\mathfrak{hol}_p(\nabla^\perp)$ has the Borel-Lichnérowicz property, i.e., we have an orthogonal decomposition*
$$N_pX = E_0 \oplus E_1 \oplus \ldots \oplus E_\ell$$
into $\mathfrak{hol}_p(\nabla^\perp)$-invariant subspaces and a corresponding decomposition
$$\mathfrak{hol}_p(\nabla^\perp) = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_\ell$$
into commuting ideals such that each $\mathfrak{h}_j \subset \mathfrak{so}(E_j, \langle\cdot,\cdot\rangle|_{E_j})$ acts weakly irreducibly on E_j and trivially on E_i for $i \neq j$.

2. *If $\mathfrak{h}_j \subset \mathfrak{so}(E_j, \langle\cdot,\cdot\rangle|_{E_j})$ acts irreducibly then it acts as one of the representations from Theorem 1.9.1.*

Proof. Since $(NX, h|_{NX}, \nabla^\perp, \pi, X)$ has good holonomy by Lemma 3.5 we may apply Proposition 1.8. $\square$

Definition 3.9. *Let $\mathfrak{g} \subset \mathfrak{so}(\mathcal{E}, h)$ for a pseudo-Euclidean vector space $(\mathcal{E}, h)$. The scalar curvature of an algebraic curvature tensor $R \in \mathcal{K}(\mathfrak{g})$ is defined as*
$$scal(R) := \sum_{k,\ell=1}^{\dim \mathcal{E}} \varepsilon_k \varepsilon_\ell h(R(e_k, e_\ell)e_\ell, e_k),$$
where $(e_1, \ldots, e_{\dim \mathcal{E}})$ is a pseudo-orthonormal basis of $(\mathcal{E}, h)$.

By the proof of Proposition 1.8
$$\mathfrak{h}_j = \mathrm{span}\{pr_{E_j} \circ \mathcal{R}_p^{\tau_\gamma^\perp}(\tau_\gamma^\perp \xi, \tau_\gamma^\perp \eta)|_{E_j} : \xi, \eta \in E_j, \gamma : [0,1] \to X,\ \gamma(0) = p\}.$$
Suppose $(E_j, \langle\cdot,\cdot\rangle|_{E_j})$ is definite. Then $\mathfrak{h}_j$ acts irreducibly and if we define $R_j := pr_{E_j} \circ$

$\mathcal{R}_p^{\tau_\gamma^\perp}(\tau_\gamma^\perp(\cdot),\tau_\gamma^\perp(\cdot))|_{E_j\times E_j\times E_j}$ then its scalar curvature is given by

$$\begin{aligned}\mathrm{scal}(R_j) &= \sum_{k,\ell=1}^{\dim E_j}\langle R_j(e_k,e_\ell)e_\ell,e_k\rangle|_{E_j}\\ &= -\frac{1}{2}\sum_{k,\ell=1}^{\dim E_j} Tr([A_{\tau_\gamma^\perp e_k},A_{\tau_\gamma^\perp e_\ell}]\circ[A_{\tau_\gamma^\perp e_k},A_{\tau_\gamma^\perp e_\ell}])\\ &= -\sum_{k,\ell=1}^{\dim E_j} B((F_{\gamma(1)}^{-1}\circ G_{\gamma(1)}\circ\tau_\gamma^\perp)(e_k,e_\ell),(F_{\gamma(1)}^{-1}\circ G_{\gamma(1)}\circ\tau_\gamma^\perp)(e_k,e_\ell)).\end{aligned}$$

If $(F_{\gamma(1)}^{-1}\circ G_{\gamma(1)}\circ\tau_\gamma^\perp)(\Lambda^2 E_j)$ is B-definite then either $\mathrm{scal}(R_j)\neq 0$ or $[A_{\tau_\gamma^\perp e_k},A_{\tau_\gamma^\perp e_\ell}]=0$ for all k,ℓ, i.e., $R_j=0$. In particular, we have $\mathrm{scal}(R_j)\neq 0\Leftrightarrow R_j\neq 0$ for a very good submanifold.

If $H_j\subset SO(E_j,\langle\cdot,\cdot\rangle|_{E_j})$ is the connected Lie subgroup having $\mathfrak{h}_j$ as its Lie algebra then H_j is compact [KN96][Appendix 5]. Hence, $(E_j,R_j\neq 0,H_j)$ is an irreducible holonomy system in the sense of Simons [Sim62] and since $\mathrm{scal}R_j\neq 0$ we can apply [Sim62][Thm. 5] in the same way as in [Olm90] to see that H_j acts as a holonomy representation of a Riemannian symmetric space.

More precisely, we have a Haar measure μ on the compact Lie group H_j and define the function

$$f: H_j\to\mathcal{K}(\mathfrak{h}_j),\qquad h\in H_j\mapsto h\cdot R_j := h\circ R_j(h^{-1}(\cdot),h^{-1}(\cdot))\circ h^{-1}.$$

Then $\tilde{R}_j := \int_{H_j} f d\mu\in\mathcal{K}(\mathfrak{h}_j)$ and $\mathrm{scal}(\tilde{R}_j)=\mathrm{scal}(R_j)\neq 0$. Since $h\cdot\tilde{R}_j=\tilde{R}_j$ by the left-translation-invariance of the Haar measure the holonomy system (E_j,R_j,H_j) is symmetric and by a construction due to Cartan [Sim62][Section 1] (E_j,R_j,H_j) corresponds to an irreducible simply connected Riemannian symmetric space whose holonomy representation is given by the action of H_j on E_j. We summarize these arguments in

Corollary 3.10. *Let $f:(X,g)\to(Y,\langle\cdot,\cdot\rangle)$ be a very good submanifold in a space of constant curvature and let $N_pX=E_0\oplus E_1\oplus\ldots\oplus E_\ell$ as well as $\mathfrak{hol}_p(\nabla^\perp)=\mathfrak{h}_1\oplus\ldots\oplus\mathfrak{h}_\ell$ be a Borel-Lichnérowicz decomposition whereas $p\in X$.*

If $\mathfrak{h}_j\subset\mathfrak{so}(E_j,\langle\cdot,\cdot\rangle|_{E_j})$ is irreducible and E_j is definite then $\mathfrak{h}_j$ acts on E_j as the holonomy representation of an irreducible Riemannian symmetric space. □

The Corollaries 3.8 and 3.10 describe the general structure of the irreducible components $\mathfrak{h}_j\subset\mathfrak{so}(E_j,\langle\cdot,\cdot\rangle|_{E_j})$ and we are left to study the remaining weakly irreducible components with non-vanishing index. At this point we remind the notation $\mathfrak{g}:=\mathfrak{g}(\mathfrak{h}):=pr_{\mathfrak{so}(p-r,q-p+r)}(\mathfrak{h})$ for a weakly irreducible subalgebra $\mathfrak{h}\subset\mathfrak{so}(p,2r-p+q)$ with index r which we introduced in Section 2.1.

Corollary 3.11. *Let $f:(X,g)\to(Y,\langle\cdot,\cdot\rangle)$ be a good submanifold in a space of constant curvature and let $\mathfrak{h}_j\subset\mathfrak{so}(E_j,\langle\cdot,\cdot\rangle|_{E_j})\cong\mathfrak{so}(p,2-p+q)$ be a weakly irreducible subalgebra with index 1 in the Borel-Lichnérowicz decomposition of $\mathfrak{hol}_p(\nabla^\perp)$. Then $\mathfrak{g}\subset\mathfrak{so}(p-1,q-$*

$p+1$) has the Borel-Lichnérowicz property. Moreover, $\mathfrak{h}_j$ is given by one of the algebras in Theorem 2.11 with $\mathfrak{g}$ acting as a Riemannian holonomy representation if $p=1$.

Proof. $\mathfrak{h}_j = \operatorname{span}\{pr_{E_j} \circ \mathcal{R}_p^{\tau_\gamma^\perp}(\tau_\gamma^\perp \xi, \tau_\gamma^\perp \eta)|_{E_j} : \xi, \eta \in E_j, \gamma : [0,1] \to X,\ \gamma(0) = p\}$ is a Berger algebra and using Corollary 2.2 we conclude the first statement. The latter follows from Corollary 2.2 and Leistner's theorem 1.9. $\square$

The normal holonomy of very good submanifolds has more structure as we see from the following

Proposition 3.12. *Let $f : (X, g) \to (Y, \langle\cdot,\cdot\rangle)$ be a very good submanifold in a space of constant curvature and let $\mathfrak{h}_j \subset \mathfrak{so}(E_j, \langle\cdot,\cdot\rangle|_{E_j}) \cong \mathfrak{so}(p, 2r-p+q)$ be a weakly irreducible subalgebra with index r in the Borel-Lichnérowicz decomposition of $\mathfrak{hol}_p(\nabla^\perp)$. Then $\mathfrak{g} := \mathfrak{g}(\mathfrak{h}_j) \subset \mathfrak{so}(p-r, q-p+r)$ is a Berger algebra. In particular, $\mathfrak{g}$ has the Borel-Lichnérowicz property.*

Proof. Using the argument following Definition 3.2 we may assume that X is simply connected as we study the normal holonomy algebra. Thus, the subspace E_j corresponds to a subbundle $\mathcal{E}$ on X. Let $\Xi_p \subset E_j$ be the isotropic r-dimensional $\mathfrak{h}_j$-invariant isotropic subspace and let Ξ be the corresponding subbundle on X. Finally, we denote a non-canonical realization of the screen bundle by S. For any $q \in X$ we have a basis $(v_1^q, \ldots, v_r^q, e_1^q, \ldots, e_q^q, w_1^q, \ldots, w_r^q)$ of $\mathcal{E}$ where $\operatorname{span}(v_1^q, \ldots, v_r^q) = \Xi_q$ and $\operatorname{span}(e_1^q, \ldots, e_q^q) = S_q$ such that

$$\langle v_i^q, v_j^q\rangle = \langle w_i^q, w_j^q\rangle = \langle v_i^q, e_j^q\rangle = \langle w_i^q, e_j^q\rangle = 0,\ \langle v_i^q, w_j^q\rangle = \delta_{ij},\ \langle e_i, e_j\rangle = \varepsilon_i \delta_{ij}.$$

Define $R^{\tau_\gamma^\perp}(\cdot,\cdot) := pr_{E_j} \circ \mathcal{R}_p^{\tau_\gamma^\perp}(\tau_\gamma^\perp(\cdot), \tau_\gamma^\perp(\cdot))|_{E_j}$. We already know that

$$\mathfrak{h}_j = \operatorname{span}\{R^{\tau_\gamma^\perp}(\xi, \eta) : \xi, \eta \in E_j, \gamma : [0,1] \to X,\ \gamma(0) = p\}$$

and from the proof of Lemma 2.1 we derive

$$\mathfrak{g} = \operatorname{span}\{\mathcal{P}_0^{R^{\tau_\gamma^\perp}}(Y_1, Y_2), \mathcal{P}_k^{R^{\tau_\gamma^\perp}}(Y_k), \mathcal{Q}_{ij}^{R^{\tau_\gamma^\perp}} : Y_\cdot \in S_p, \gamma : [0,1] \to X,\ \gamma(0) = p\}.$$

Once we have shown $\mathcal{P}_k^{R^{\tau_\gamma^\perp}}(Y_k), \mathcal{Q}_{ij}^{R^{\tau_\gamma^\perp}} \in \operatorname{span}\{\mathcal{P}_0^{R^{\tau_\gamma^\perp}}(Y_1, Y_2) : Y_\cdot \in S_p\}$ the statement follows as $\langle \mathcal{P}_0^{R^{\tau_\gamma^\perp}}(Y_1, Y_2) Y_3, Y_4\rangle = \langle \mathcal{R}_{\gamma(1)}(\tau_\gamma^\perp Y_1, \tau_\gamma^\perp Y_2)\tau_\gamma^\perp Y_3, \tau_\gamma^\perp Y_4\rangle$, i.e., $\mathcal{P}_0^{R^{\tau_\gamma^\perp}}$ is an algebraic curvature tensor on $\mathcal{E}$ by Lemma 3.3. For any $Y_\cdot \in S_p$ we have $\tau_\gamma^\perp Y_\cdot := V_\cdot + \tilde{Y}_\cdot$ where $\tilde{Y}_\cdot \in S_{\gamma(1)}$ and $V \in \Xi_{\gamma(1)}$. Therefore,

$$\begin{aligned}
\langle \mathcal{Q}_{ij}^{R^{\tau_\gamma^\perp}} Y_3, Y_4\rangle &= \langle \mathcal{R}_{\gamma(1)}(\tau_\gamma^\perp w_i, \tau_\gamma^\perp w_j)\tau_\gamma^\perp Y_3, \tau_\gamma^\perp Y_4\rangle = \kappa([A_{\tau_\gamma^\perp w_i}, A_{\tau_\gamma^\perp w_j}], [A_{\tau_\gamma^\perp Y_3}, A_{\tau_\gamma^\perp Y_4}]) \\
&= \kappa([A_{\tau_\gamma^\perp w_i}, A_{\tau_\gamma^\perp w_j}], [A_{V_3 + \tilde{Y}_3}, A_{V_4 + \tilde{Y}_4}]) \\
&= \kappa([A_{\tau_\gamma^\perp w_i}, A_{\tau_\gamma^\perp w_j}], [A_{V_3}, A_{V_4}]) + \kappa([A_{\tau_\gamma^\perp w_i}, A_{\tau_\gamma^\perp w_j}], [A_{V_3}, A_{\tilde{Y}_4}]) \\
&\quad + \kappa([A_{\tau_\gamma^\perp w_i}, A_{\tau_\gamma^\perp w_j}], [A_{\tilde{Y}_3}, A_{V_4}]) + \kappa([A_{\tau_\gamma^\perp w_i}, A_{\tau_\gamma^\perp w_j}], [A_{\tilde{Y}_3}, A_{\tilde{Y}_4}])
\end{aligned}$$

and

$$\begin{aligned}\langle \mathcal{P}_k^{R^{\tau_\gamma^\perp}}(Y_k)Y_3, Y_4\rangle &= \kappa([A_{\tau_\gamma^\perp w_k}, A_{\tau_\gamma^\perp Y_k}], [A_{V_3}, A_{V_4}]) + \kappa([A_{\tau_\gamma^\perp w_k}, A_{\tau_\gamma^\perp Y_k}], [A_{V_3}, A_{\tilde{Y}_4}]) \\ &\quad + \kappa([A_{\tau_\gamma^\perp w_k}, A_{\tau_\gamma^\perp Y_k}], [A_{\tilde{Y}_3}, A_{V_4}]) + \kappa([A_{\tau_\gamma^\perp w_k}, A_{\tau_\gamma^\perp Y_k}], [A_{\tilde{Y}_3}, A_{\tilde{Y}_4}]).\end{aligned}$$

Since $V_. \in \Xi_{\gamma(1)}$ the Ricci equation implies for any $\tilde{V} \in \Xi_{\gamma(1)} \oplus S_{\gamma(1)}$

$$0 = \langle R^\perp(\cdot,\cdot)V_., \tilde{V}\rangle = \langle [A_{V_.}, A_{\tilde{V}}](\cdot), \cdot\rangle,$$

i.e., $[A_{V_.}, A_{\tilde{V}}] = 0$. Since X is a very good submanifold $G_q(\Lambda^2 S_q)$ is a non-degenerate subspace in $(\mathfrak{so}(T_qX, g), \kappa)$ for all $q \in X$. Let $(b_1, \dots, b_m)$ be a κ-pseudo-orthonormal basis of $G_{\gamma(1)}(\Lambda^2 S_{\gamma(1)})$ and $b_i =: \sum_{k<\ell} \lambda^i_{k\ell} G_{\gamma(1)}(e^q_k \wedge e^q_\ell)$. We derive

$$\begin{aligned}\langle \mathcal{Q}_{ij}^{R^{\tau_\gamma^\perp}} Y_3, Y_4\rangle &= \sum_{n=1}^m \varepsilon_n \kappa([A_{\tau_\gamma^\perp w_i}, A_{\tau_\gamma^\perp w_j}], b_n)\kappa(b_n, [A_{\tilde{Y}_3}, A_{\tilde{Y}_4}]) \\ &= \sum_{k<\ell} \varepsilon_n \lambda^n_{k\ell} \kappa([A_{\tau_\gamma^\perp w_i}, A_{\tau_\gamma^\perp w_j}], b_n)\kappa([A_{e^q_k}, A_{e^q_\ell}], [A_{\tilde{Y}_3}, A_{\tilde{Y}_4}])\end{aligned}$$

and

$$\langle \mathcal{P}_i^{R^{\tau_\gamma^\perp}}(Y_i)Y_3, Y_4\rangle = \sum_{k<\ell} \varepsilon_n \lambda^n_{k\ell} \kappa([A_{\tau_\gamma^\perp w_i}, A_{\tau_\gamma^\perp Y_i}], b_n)\kappa([A_{e^q_k}, A_{e^q_\ell}], [A_{\tilde{Y}_3}, A_{\tilde{Y}_4}]).$$

We can find $\check{e}_. \in S_p$ such that $pr_{S_{\gamma(1)}}(\tau_\gamma^\perp \check{e}_.) = e^q_.$ and using the Ricci equation once more we conclude $\kappa([A_{e^q_k}, A_{e^q_\ell}], [A_{\tilde{Y}_3}, A_{\tilde{Y}_4}]) = \langle \mathcal{P}_0^{R^{\tau_\gamma^\perp}}(\check{e}_k, \check{e}_\ell)Y_3, Y_4\rangle$. □

Remark 3.13. The proof of Proposition 3.12 still works for good submanifolds once we know that for all $q \in X$ the subspace $G_q(\Lambda^2 S_q) \subset \mathfrak{so}(T_qX, g)$ is non-degenerate w.r.t. the Killing form.

Good submanifolds for which this property fails can be seen as the extrinsic counterpart to manifolds with *lightlike hypersurface curvature* which have been defined in [Lei06][Definition 5.2].

An identification of good submanifolds for which $G_q(\Lambda^2 S_q) \subset \mathfrak{so}(T_qX, g)$ is definite will be given in Prop. 3.16. □

Theorem 3.14. *Let $f : (X, g) \to (Y, \langle\cdot,\cdot\rangle)$ be a simply connected very good submanifold in a space of constant curvature. Then*

- $(NX, h|_{NX}, \nabla^\perp, \pi, X)$ *admits a complete screen tree whose non-trivial leaves are given by Theorem 1.9,*
- *any non-trivial leaf given by a representation on a definite space acts as the holonomy representation of an irreducible Riemannian symmetric space.*

Proof. We give a proof by induction as indicated in the following diagram

$$\begin{array}{ccccc} & & \swarrow & & \\ & & \mathfrak{h}_i^{[\ell]} & & \\ & \swarrow & \downarrow & \searrow & \\ \mathfrak{h}_1^{[\ell+1]} & \dots & \mathfrak{h}_j^{[\ell+1]} & \dots & \mathfrak{h}_{n_{\ell+1}}^{[\ell+1]} \\ \swarrow & & \downarrow & & \searrow \\ \mathfrak{h}_1^{[\ell+2]} & \dots & \mathfrak{h}_k^{[\ell+2]} & \dots & \mathfrak{h}_{n_{\ell+2}}^{[\ell+2]} \\ & \swarrow & & & \end{array}$$

More precisely, we write $\mathfrak{h}_1^{[0]} := \mathfrak{hol}_p(\nabla^\perp)$. The irreducible subalgebras in the Borel-Lichnérowicz decomposition of $\mathfrak{hol}_p(\nabla^\perp)$ are children of $\mathfrak{h}_1^{[0]}$ and if $\mathfrak{h}_j \subset \mathfrak{so}(E_j, \langle\cdot,\cdot\rangle|_{E_j}) \cong \mathfrak{so}(p, 2r-p+q)$ is a weakly irreducible subalgebra with index r in the Borel-Lichnérowicz decomposition of $\mathfrak{hol}_p(\nabla^\perp)$ then Prop. 3.12 implies the existence of a possibly trivial child

$$\mathfrak{h}_j^{[1]} = \mathrm{span}\{\mathcal{P}_0^{R^{\tau_\gamma^\perp}}(Y_1, Y_2) : Y_\cdot \in S_j^{[1]}|_p, \gamma : [0,1] \to X,\ \gamma(0) = p\},$$

where $S_j^{[1]}|_p$ is the fiber of the screen bundle associated to $\mathfrak{h}_j$ at $p \in X$. In particular, $\mathfrak{h}_j^{[1]}$ can be computed by restriction of $R^{\tau_\gamma^\perp}$ to $S_j^{[1]}|_p$. We conclude the statement for the irreducible children from Corollary 3.10.

For the inductive step we consider the algebra $\mathfrak{h}_i^{[\ell]}$. By induction hypothesis

$$\mathfrak{h}_i^{[\ell]} = \mathrm{span}\{R_{[\ell],i}^{\tau_\gamma^\perp}(Y_1, Y_2) : Y_\cdot \in S_i^{[\ell]}|_p, \gamma : [0,1] \to X,\ \gamma(0) = p\},$$

where we define $S_i^{[\ell]}|_p$ to be the fiber of the screen bundle associated to a weakly irreducible component in the Borel-Lichnérowicz decomposition of $\mathfrak{h}_j^{[\ell-1]}$ and $R_{[\ell],i}^{\tau_\gamma^\perp}(\cdot,\cdot) := pr_{S_i^{[\ell]}} \circ \mathcal{R}_p^{\tau_\gamma^\perp}(\tau_\gamma^\perp(\cdot), \tau_\gamma^\perp(\cdot))|_{S_i^{[\ell]}}$. Hence, $\mathfrak{h}_i^{[\ell]}$ is a Berger algebra and it admits a Borel-Lichnérowicz decomposition. Let $\mathfrak{h}_j$ be a weakly irreducible subalgebra in this decomposition acting on $S_i^{[\ell]}|_p(j) \subset S_i^{[\ell]}|_p$.

If $\mathfrak{h}_j$ acts irreducibly then $\mathfrak{h}_j^{[\ell+1]} := \mathfrak{h}_j$ is given by the list in Theorem 1.9. If $\mathfrak{h}_j$ additionally acts on a definite space $S_i^{[\ell]}|_p(j)$ then we have a non-trivial algebraic curvature tensor

$$R_j^\gamma := pr_{S_i^{[\ell]}|_p(j)} \circ R_{[\ell],i}^{\tau_\gamma^\perp}|_{S_i^{[\ell]}|_p(j)} \in \mathcal{K}(\mathfrak{h}_j^{[\ell+1]})$$

whose scalar curvature is non-vanishing since for $d := \dim S_i^{[\ell]}|_p(j)$

$$\begin{aligned}\mathrm{scal}(R_j^\gamma) &= \frac{1}{2}\sum_{\alpha,\beta=1}^{d} Tr([A_{\tau_\gamma^\perp e_\alpha}, A_{\tau_\gamma^\perp e_\beta}] \circ [A_{\tau_\gamma^\perp e_\alpha}, A_{\tau_\gamma^\perp e_\beta}]) \\ &= \sum_{\alpha,\beta=1}^{d} B((F_{\gamma(1)}^{-1} \circ G_{\gamma(1)})(\tau_\gamma^\perp e_\alpha, \tau_\gamma^\perp e_\beta), (F_{\gamma(1)}^{-1} \circ G_{\gamma(1)})(\tau_\gamma^\perp e_\alpha, \tau_\gamma^\perp e_\beta))\end{aligned}$$

and $(F_{\gamma(1)}^{-1} \circ G_{\gamma(1)} \circ \tau_\gamma^\perp)(\Lambda^2 S_i^{[\ell]}|_p(j))$ is B-definite. Hence, $\mathfrak{h}_j^{[\ell+1]}$ acts on $S_i^{[\ell]}|_p(j)$ as the holonomy representation of an irreducible Riemannian symmetric space by the same argument as in Corollary 3.10.

Finally, we assume $\mathfrak{h}_j$ acts with index r. Thus, there is a possibly trivial[3] associated screen representation $\mathfrak{h}_j^{[\ell+1]} \subset \mathfrak{so}(S_j^{[\ell+1]}|_p, \langle\cdot,\cdot\rangle|_{S_j^{[\ell+1]}|_p})$ and general theory implies

$$\mathfrak{h}_j^{[\ell+1]} = \mathrm{span}\{\mathcal{P}_0^{R_j^\gamma}(Y_1, Y_2), \mathcal{P}_k^{R_j^\gamma}(Y_k), \mathcal{Q}_{\alpha\beta}^{R_j^\gamma} : Y_\cdot \in S_j^{[\ell+1]}|_p,\ \gamma(0) = p\}.$$

Since $R_{[\ell+1],j}^{\tau_\gamma^\perp} = \mathcal{P}_0^{R_j^\gamma}$ we conclude the inductive step once we have shown $\mathcal{P}_k^{R_j^\gamma}(Y_k), \mathcal{Q}_{\alpha\beta}^{R_j^\gamma} \in \mathrm{span}\{\mathcal{P}_0^{R_j^\gamma}(Y_1, Y_2) : Y_\cdot \in S_j^{[\ell+1]}|_p\}$. In this case all non-trivial vertices are Berger algebras by induction and $(NX, h|_{NX}, \nabla^\perp, \pi, X)$ admits a complete screen tree whose non-trivial leaves are as indicated. In order to show the relation between $\mathcal{P}_k^{R_j^\gamma}$, $\mathcal{Q}_{\alpha\beta}^{R_j^\gamma}$ and $\mathcal{P}_0^{R_j^\gamma}$ we can apply the same arguments as in the proof of Prop. 3.12 with the obvious substitutions since for any very good submanifold $G_q(\Lambda^2 S_j^{[\ell+1]}|_q)$ is κ-non-degenerate for all $q \in X$. $\square$

3.2 Tubes Along Subbundles

Given the results from the last section and Theorem 2.11 we derive a coarse classification of the restricted normal holonomy representations of spacelike submanifolds in Minkowski space. Because of the non-existence of a geometric de Rham decomposition theorem 1.14 for the holonomy of the normal screen bundle[4] we provide a construction making the normal screen holonomy irreducible.

Let $f : (X, g) \to (Y, h)$ be an embedded submanifold. For any $v \in N_pX$ let γ_v be the (Y, h)-geodesic such that $\gamma_v(0) = p$ and $\dot\gamma_v(0) = v$. It is well known that there exists an open neighborhood $U \subset NX$ of the zero section in NX on which we can define the normal exponential map

$$\exp^\perp : U \to Y \qquad v \mapsto \gamma_v(1)$$

[3]By definition of the screen tree we attach a trivial vertex to $\mathfrak{h}_j$ if $\dim S_j^{[\ell+1]}|_p = 0$. The theorem does not exclude the case in which $\dim S_i^{[\ell]}|_p(j) = 2r$ and $\mathfrak{h}_j$ acts weakly irreducibly with index r on $S_i^{[\ell]}|_p(j)$.

[4]Similarly, there is no extrinsic counterpart to the geometric de Rham theorem for normal holonomies, i.e., we may not assume the normal holonomy to act weakly irreducibly. E.g., we can consider spacelike submanifolds in de Sitter space as submanifolds in Minkowski space.

and by shrinking U we may assume $\exp^\perp$ to be a diffeomorphism of U onto its image. As usual we call $\exp^\perp(U)$ a tubular neighborhood of $f(X) \subset Y$. For any subbundle $\iota : E \hookrightarrow NX$ we have an embedding $(\exp^\perp \circ \iota)|_{\iota^{-1}(U)}$.

Definition 3.15. *Let $f : (X, g) \to (Y, h)$ be an embedded submanifold and $U \subset NX$ a neighborhood of the zero section on which $\exp^\perp$ is a diffeomorphism onto an open neighborhood of $f(X) \subset Y$. For a subbundle $\iota : E \to NX$ the image*

$$\mathcal{U}_E := (\exp^\perp \circ \iota)(\iota^{-1}(U)) \subset Y$$

is called the tube along E in (Y, h).

Consider a smooth curve $\beta_s : (-\varepsilon, +\varepsilon) \to \mathcal{U}_E$ and its associated curve $\alpha_s : (-\varepsilon, +\varepsilon) \to X$. Then we have a normal vector field V_s on α_s such that $\gamma_{V_s}(1) = \beta_s$, where γ_{V_s} is the (Y, h)-geodesic satisfying $\gamma_{V_s}(0) = \alpha_s$ and $\dot\gamma_{V_s}(0) = V_s$. Since γ_{V_s} is a (Y, h)-geodesic for all $s \in (-\varepsilon, +\varepsilon)$ we derive the following geodesic variation

$$\psi : [0, 1] \times (-\varepsilon, +\varepsilon) \to Y \qquad (t, s) \mapsto \gamma_{V_s}(t).$$

Hence, the variation vector field $W_t := \frac{\partial}{\partial s}\psi(t, \cdot)|_{s=0}$ of ψ is a Jacobi field on γ_{V_0} and $\psi(0, s) = \alpha_s$ as well as $W_0 = \dot\alpha_0$. In particular, $\dot\beta_0 = W_1$ and $\frac{\nabla^Y}{dt} W_t|_{t=0} = \frac{\nabla^Y}{ds}\frac{\partial}{\partial t}\psi(t, s)|_{t,s=0} = \frac{\nabla^Y}{ds} V_s|_{s=0} = -A^f_{V_0}(\dot\alpha_0) + \nabla^{\perp,f}_{\dot\alpha_0} V_s$. Suppose $(Y, h) = (\mathbb{R}^{r+s}, \langle\cdot,\cdot\rangle_{r,s})$. In this case,

$$\begin{aligned} W_t &= \tau^{\nabla^{\mathbb{R}^{r,s}}}_{\gamma_{V_0}}(\dot\alpha_0)(t) + t \cdot \tau^{\nabla^{\mathbb{R}^{r,s}}}_{\gamma_{V_0}}(-A^f_{V_0}(\dot\alpha_0) + \nabla^{\perp,f}_{\dot\alpha_0} V_s)(t), \text{ i.e.,} \\ \dot\beta_0 &= \tau^{\nabla^{\mathbb{R}^{r,s}}}_{\gamma_{V_0}}(\dot\alpha_0 - A^f_{V_0}(\dot\alpha_0) + \nabla^{\perp,f}_{\dot\alpha_0} V_s). \end{aligned}$$

Let $p \in X$ and $q = \exp^\perp(V)$ for some $V \in E_p \cap U$. For any $w \in T_pX$ let $\alpha^w : (-\varepsilon, +\varepsilon) \to X$ such that $\dot\alpha^w(0) = w$. We conclude

$$T_q\mathcal{U}_E = \{\tau^{\nabla^{\mathbb{R}^{r,s}}}_{\gamma_V}(w - A^f_V(w) + \nabla^{\perp,f}_w V_s) : w \in T_pX,\ V_s \in \Gamma(\alpha^w, E), V_0 = V\}.$$

Since parallel displacement preserves orthogonality we derive

$$N_q\mathcal{U}_E = \{\tau^{\nabla^{\mathbb{R}^{r,s}}}_{\gamma_V}(v) : v \in T_p\mathbb{R}^{r+s},\ v \perp w - A^f_V(w) + \nabla^{\perp,f}_w V_s\}.$$

Consider the linear map $F_{V_0} : T_pX \to T_pX$ such that $F_{V_0}(w) := w - A^f_{V_0}(w)$. If F_{V_0} is not surjective we have $0 \neq w \in T_pX$ such that $F_{V_0}(w) = 0$. In this case, we choose $V_s \in \Gamma(\alpha^w, NX)$ to be $\nabla^{\perp,f}$-parallel along α^w. Hence, we have $W_0 = w \neq 0$ and $W_1 = \tau^{\nabla^{\mathbb{R}^{r,s}}}_{\gamma_{V_0}}(F_{V_0}(w)) = 0$ for the associated Jacobi field on γ_{V_0}. Thus, $\gamma_{V_0}(1)$ is a focal point of X along γ_{V_0}, i.e., $\exp^\perp$ is singular at V_0 implying a contradiction.[5]

For a submanifold $f : (X, g) \to (\mathbb{R}^{r+s}, \langle\cdot,\cdot\rangle_{r,s})$ whose normal holonomy algebra acts

[5] Using this observation it is still hard to compute $T_q\mathcal{U}_E$ since we may not choose $V_s \in \Gamma(\alpha^w, E)$ to be $\nabla^{\perp,f}$-parallel if E_p is not invariant under the action of $\mathfrak{hol}^{loc}_p(\nabla^{\perp,f})$.

weakly irreducibly with index 1 there is a globally defined $\nabla^{\perp}$-parallel isotropic subbundle Ξ on X by Lemma 2.10. In particular, the normal screen bundle is defined. Moreover, an extension of X along $S^{\perp}$ provides a geometric tool to identify good submanifolds with extrinsic lightlike hypersurface curvature as we can see from

Proposition 3.16. *Let $f : (X, g) \to (\mathbb{R}^{r+s}, \langle\cdot,\cdot\rangle_{r,s})$ be an embedded good submanifold whose normal holonomy acts weakly irreducibly with index 1. If S is a non-canonical realization of the normal screen bundle then there is an open subset $\tilde{X} \supset X$ of $\mathcal{U}_{S^{\perp}} \subset (\mathbb{R}^{r+s}, \langle\cdot,\cdot\rangle_{r,s})$ which is a very good submanifold if and only if for all $p \in X$ the subspace $G_p(\Lambda^2 S_p) \subset \mathfrak{so}(T_pX, g)$ is definite w.r.t. the Killing form.*

Moreover, the normal holonomy representation of $\tilde{X} \subset (\mathbb{R}^{r+s}, \langle\cdot,\cdot\rangle_{r,s})$ contains that of the normal screen bundle.

Proof. Consider a non-canonical realization S of the screen bundle and the tube $\mathcal{U}_{S^{\perp}}$ along $S^{\perp}$ in $\mathbb{R}^{r,s}$. Since $\Xi \subset S^{\perp}$ there is an open subset $p \in U \subset X$ and a local frame $(V, Z, e_1, \ldots, e_d)$ of $T_p\mathcal{U}_{S^{\perp}}|_U$ around p where $(e_1|_p, \ldots, e_d|_p)$ is a basis of $T_pX \subset T_p\mathcal{U}_{S^{\perp}}$ such that $\langle e_i, e_j\rangle = \varepsilon_i\delta_{ij}$ and $V, Z \in \Gamma(U, S^{\perp})$ such that $V \in \Xi$, $\langle V, Z\rangle = 1$, $\langle Z, Z\rangle = 0$ and $\langle V, e_i\rangle = \langle Z, e_i\rangle = 0$. Moreover, we have $\nabla^{\perp,f}_{\cdot} V \in \mathbb{R}\cdot V$. Let $(E_1, \ldots, E_m)$ with $E_{\cdot} \in \Gamma(U \subset X, S)$ be a local pseudo-orthonormal frame of S around p and $w \in T_pX$. For the shape operators of $\mathcal{U}_E$ at p we compute

$$\begin{aligned} A^{\mathcal{U}_{S^{\perp}}}_{E_k}(w) &= -pr_{T_p\mathcal{U}_{S^{\perp}}} \circ \nabla^{\mathbb{R}^{r,s}}_w E_k \\ &= -pr_{T_p\mathcal{U}_{S^{\perp}}}(pr_{T_pX} \circ \nabla^{\mathbb{R}^{r,s}}_w E_k + pr_{N_pX} \circ \nabla^{\mathbb{R}^{r,s}}_w E_k) \\ &= -pr_{T_p\mathcal{U}_{S^{\perp}}}(-A^f_{E_k}(w) + \nabla^{\perp,f}_w E_k) \\ &= A^f_{E_k}(w) - pr_{S^{\perp}} \circ \nabla^{\perp,f}_w E_k \\ &= A^f_{E_k}(w) - pr_{\Xi} \circ \nabla^{\perp,f}_w E_k \end{aligned}$$

since $\nabla^{\perp,f}_w(E_k \in \Xi^{\perp}) \in \Xi^{\perp}$. We define $\nabla^{\Xi}_w E_k := pr_{\Xi} \circ \nabla^{\perp,f}_w E_k$. Thus, $\langle A^{\mathcal{U}_{S^{\perp}}}_{E_k} e_i, e_j\rangle = \langle A^f_{E_k} e_i, e_j\rangle$ as well as $\langle A^{\mathcal{U}_{S^{\perp}}}_{E_k} V, e_j\rangle = \langle A^{\mathcal{U}_{S^{\perp}}}_{E_k} e_j, V\rangle = 0$ and $\langle A^{\mathcal{U}_{S^{\perp}}}_{E_k} Z, e_j\rangle = \langle A^{\mathcal{U}_{S^{\perp}}}_{E_k} e_j, Z\rangle = -\nabla^{\Xi}_{e_j} E_k$.

In order to compute $A^{\mathcal{U}_{S^{\perp}}}_{E_k}(V)$ we have to extend $E_k|_p$ to a section $\tilde{E}_k \in \Gamma(\gamma_V, N\mathcal{U}_{S^{\perp}})$ along the geodesic γ_V with $\gamma_V(0) = p$ and $\dot{\gamma}_V(0) = V_p$. We have

$$T_{\gamma_V}\mathcal{U}_{S^{\perp}} = \{\tau^{\nabla^{\mathbb{R}^{r,s}}}_{\gamma_V}(w - A^f_V(w) + \nabla^{\perp,f}_w W_s) : W_s \in \Gamma(\alpha^{w \in T_pX}, S^{\perp}), W_0 = V_p\}$$

and since $\nabla^{\perp,f} V \in \mathbb{R}\cdot V$ we have $\xi_t \in \mathbb{R}\cdot V_{\alpha^w_t}$ for the $\nabla^{\perp,f}$-parallel displacement ξ_t of V_p along α^w, i.e., $\xi_t \in \Gamma(\alpha^w, S^{\perp})$. Using $W_t := \xi_t$ we derive $\tau^{\nabla^{\mathbb{R}^{r,s}}}_{\gamma_V(t)}(w - A^f_V(w)) \in T_{\gamma_V(t)}\mathcal{U}_{S^{\perp}}$, i.e., $\tau^{\nabla^{\mathbb{R}^{r,s}}}_{\gamma_V(t)}(T_pX) \in T_{\gamma_V(t)}\mathcal{U}_{S^{\perp}}$. Moreover, using $w = 0$ and $W_s = V + s\cdot Z$ resp. $W_s = (s+1)V$ we have $\tau^{\nabla^{\mathbb{R}^{r,s}}}_{\gamma_V(t)}(Z) \in T_{\gamma_V(t)}\mathcal{U}_{S^{\perp}}$ resp. $\tau^{\nabla^{\mathbb{R}^{r,s}}}_{\gamma_V(t)}(V) \in T_{\gamma_V(t)}\mathcal{U}_{S^{\perp}}$. We conclude

$$\tilde{E}_k|_{\gamma_V(t)} := \tau_{\gamma_V(t)}^{\nabla^{\mathbb{R}^{r,s}}}(E_k) \in N_{\gamma_V(t)}\mathcal{U}_{S^\perp} \text{ and } A_{E_k}^{\mathcal{U}_{S^\perp}}(V) = -pr_{T_p\mathcal{U}_{S^\perp}} \circ \nabla_V^{\mathbb{R}^{r,s}} \tilde{E}_k = 0.$$

Hence, the matrix of $A_{E_k}^{\mathcal{U}_{S^\perp}}$ in the basis $(V, Z, e_1, \dots, e_d)$ is given by

$$A_{E_k}^{\mathcal{U}_{S^\perp}} = \begin{pmatrix} 0 & *_k & -\nabla_{e_1}^{\Xi} E_k & \dots & -\nabla_{e_d}^{\Xi} E_k \\ 0 & 0 & 0 & \dots & 0 \\ 0 & -\nabla_{e_1}^{\Xi} E_k & & & \\ \vdots & \vdots & & A_{E_k}^f & \\ 0 & -\nabla_{e_d}^{\Xi} E_k & & & \end{pmatrix},$$

where $*_k := pr_V(A_{E_k}^{\mathcal{U}_{S^\perp}}(Z)) = \langle A_{E_k}^{\mathcal{U}_{S^\perp}}(Z), Z\rangle$. Moreover,

$$A_{E_k}^{\mathcal{U}_{S^\perp}} A_{E_\ell}^{\mathcal{U}_{S^\perp}} = \begin{pmatrix} 0 & \sum_{i=1}^d \nabla_{e_i}^{\Xi} E_k \nabla_{e_i}^{\Xi} E_\ell & \mathcal{A}_1^{k,\ell} & \dots & \mathcal{A}_d^{k,\ell} \\ 0 & 0 & 0 & \dots & 0 \\ 0 & \mathcal{B}_1^{k,\ell} & & & \\ \vdots & \vdots & & A_{E_k}^f A_{E_\ell}^f & \\ 0 & \mathcal{B}_d^{k,\ell} & & & \end{pmatrix}, \text{ where}$$

$\mathcal{A}_j^{k,\ell} := -\sum_{i=1}^d \varepsilon_i \langle A_{E_\ell}^f e_j, e_i\rangle \nabla_{e_i}^{\Xi} E_k$ and $\mathcal{B}_j^{k,\ell} := -\sum_{i=1}^d \varepsilon_j \langle A_{E_k}^f e_i, e_j\rangle \nabla_{e_i}^{\Xi} E_\ell$. We conclude

$$[A_{E_k}^{\mathcal{U}_{S^\perp}}, A_{E_\ell}^{\mathcal{U}_{S^\perp}}] = \begin{pmatrix} 0 & 0 & \mathcal{A}_1^{k,\ell} - \mathcal{A}_1^{\ell,k} & \dots & \mathcal{A}_d^{k,\ell} - \mathcal{A}_d^{\ell,k} \\ 0 & 0 & 0 & \dots & 0 \\ 0 & \mathcal{B}_1^{k,\ell} - \mathcal{B}_1^{\ell,k} & & & \\ \vdots & \vdots & & [A_{E_k}^f, A_{E_\ell}^f] & \\ 0 & \mathcal{B}_d^{k,\ell} - \mathcal{B}_d^{\ell,k} & & & \end{pmatrix},$$

i.e.,

$$[A_{E_k}^{\mathcal{U}_{S^\perp}}, A_{E_\ell}^{\mathcal{U}_{S^\perp}}] \cdot [A_{E_k}^{\mathcal{U}_{S^\perp}}, A_{E_\ell}^{\mathcal{U}_{S^\perp}}] = \begin{pmatrix} 0 & *_0 & *_{a1} & \dots & *_{ad} \\ 0 & 0 & 0 & \dots & 0 \\ 0 & *_{b1} & & & \\ \vdots & \vdots & & [A_{E_k}^f, A_{E_\ell}^f] \cdot [A_{E_k}^f, A_{E_\ell}^f] & \\ 0 & *_{bd} & & & \end{pmatrix}.$$

Thus, $Tr([A_{E_k}^{\mathcal{U}_{S^\perp}}, A_{E_\ell}^{\mathcal{U}_{S^\perp}}] \cdot [A_{E_k}^{\mathcal{U}_{S^\perp}}, A_{E_\ell}^{\mathcal{U}_{S^\perp}}]) = Tr([A_{E_k}^f, A_{E_\ell}^f] \cdot [A_{E_k}^f, A_{E_\ell}^f])$.

As we have seen in Appendix 3.2 $G_p(\Lambda^2 S_p) \subset \mathfrak{so}(T_pX, g)$ is definite if and only if $Tr([A_{E_k}^f, A_{E_\ell}^f] \cdot [A_{E_k}^f, A_{E_\ell}^f]) > 0$ or < 0 for all k, ℓ. However, by continuity there is an open subset $\tilde{X} \subset \mathcal{U}_{S^\perp}$ with $X \subset \tilde{X}$ if and only if $Tr([A_{E_k}^{\mathcal{U}_{S^\perp}}, A_{E_\ell}^{\mathcal{U}_{S^\perp}}] \cdot [A_{E_k}^{\mathcal{U}_{S^\perp}}, A_{E_\ell}^{\mathcal{U}_{S^\perp}}]) > 0$ or < 0 for all k, ℓ.

It remains to show the holonomy statement. Let $\alpha : [0,1] \to X$ be a curve such that $\alpha(0) = \alpha(1) = p \in X$ and $\xi \in \Gamma(\alpha, S)$ be the parallel displacement of some $\xi_0 \in S_p$ w.r.t. $pr_S \circ \nabla^{\perp,f}_{\cdot}$. We conclude that ξ is parallel w.r.t. $\nabla^{\perp,\mathcal{U}_{S^\perp}}_{\cdot} = pr_{N\mathcal{U}_{S^\perp}} \circ \nabla^{\mathbb{R}^{r,s}}_{\cdot}$ since α is a curve in X, i.e., $N_{\alpha(t)}\mathcal{U}_{S^\perp} = S_{\alpha(t)}$ for all $t \in [0,1]$. Hence, $Hol(pr_S \circ \nabla^{\perp,f}) \subset Hol(\nabla^{\perp,\mathcal{U}_{S^\perp}})$. □

Let $\gamma : [0,1] \to \mathbb{R}^{1,n+1}$ be a lightlike curve in Minkowski space such that $\dot{\gamma}_t \neq 0$ for all $t \in [0,1]$. We can find a lightlike vector field $Z \in \Gamma(\gamma, T\mathbb{R}^{1,n+1})$ such that $\langle \dot{\gamma}_t, Z_t \rangle = 1$ for all $t \in [0,1]$ and define the vector bundles $S := \operatorname{span}\{\dot{\gamma}, Z\}^\perp$ and $W := \operatorname{span}\{Z, Y_1, \ldots, Y_{m-1}\}$ over γ where $(Y_1, \ldots, Y_n)$ is a given orthonormal frame of S along γ and $m \leq n$. Moreover, we consider the "tube along W" in $\mathbb{R}^{1,n+1}$, i.e., the immersion

$$\Psi_W : [0,1] \times U \subset \mathbb{R}^m \to \mathbb{R}^{1,n+1}, \qquad (t, x^0, \ldots, x^{m-1}) \mapsto \gamma_t + x^0 Z_t + \sum_{i=1}^{m-1} x^i Y_i.$$

This way, we derive an $(m+1)$-dimensional Lorentzian submanifold X_W of $\mathbb{R}^{1,n+1}$ to which we refer as the $(m+1)$-dimensional null scroll in $\mathbb{R}^{1,n+1}$ associated to W. These submanifolds have been studied in [BE04] and we use them for the following warning.

Corollary 3.17. *Let X be an $(m+1)$-dimensional null scroll in $\mathbb{R}^{1,n+1}$. Then X is a good submanifold if and only if its normal bundle is flat.*

Proof. It is shown in [BE04] that given a local frame $(V, Z, Y_1, \ldots, Y_{m-1})$ of TX around $p \in X$ such that $\langle V, Z \rangle = -1$, $\langle V, V \rangle = 0$ and an orthonormal frame $(\xi_1, \ldots, \xi_{n-m-1})$ of NX around p we have for $1 \leq j < n - m$

$$A_{\xi_j} = -\begin{pmatrix} a^j & b^j_{00} & c^j_1 & \cdots & c^j_{m-1} \\ 0 & b^j_{10} & 0 & \cdots & 0 \\ 0 & b^j_{11} & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & b^j_{1(m-1)} & 0 & \cdots & 0 \end{pmatrix}.$$

Hence,

$$[A_{\xi_i}, A_{\xi_j}] = \begin{pmatrix} 0 & * & a^i c^j_1 - a^j c^i_1 & \cdots & a^i c^j_{m-1} - a^j c^i_{m-1} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & b^i_{11} b^j_{01} - b^j_{11} b^i_{01} & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & b^i_{1(m-1)} b^j_{01} - b^j_{1(m-1)} b^i_{01} & 0 & \cdots & 0 \end{pmatrix}$$

and

$$Tr([A_{\xi_i}, A_{\xi_j}] \cdot [A_{\xi_\alpha}, A_{\xi_\beta}]) = Tr(\begin{pmatrix} 0 & * & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}) = 0.$$

In this case, X is very good if and only if $[A_{\xi_i}, A_{\xi_j}] = 0$ for all $1 \leq i, j \leq n-m-1$ which is equivalent to X having a flat normal bundle. $\square$

Proposition 3.18. *Let $f : (X, g) \to (\mathbb{R}^{r+s}, \langle\cdot,\cdot\rangle_{r,s})$ be an embedded good submanifold whose Borel-Lichnérowicz decomposition is given by*

$$N_pX = E_0 \oplus E_1 \oplus \ldots \oplus E_\ell \quad \textit{and} \quad \mathfrak{hol}_p(\nabla^\perp) = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_\ell$$

such that $\mathfrak{h}_2, \ldots, \mathfrak{h}_\ell$ act irreducibly, E_0 is definite and $\mathfrak{so}(1,1) \neq \mathfrak{h}_1 \subset \mathfrak{so}(E_1, \langle\cdot,\cdot\rangle|_{E_1})$ acts weakly irreducibly with index 1. Then a non-canonical realization S of the normal screen bundle is defined and $\mathfrak{h}_1$ leaves a lightlike vector invariant if $\mathcal{U}_S \subset (\mathbb{R}^{r+s}, \langle\cdot,\cdot\rangle_{r,s})$ has flat normal bundle.

Proof. Using Lemma 2.10 and the holonomy principle we derive a $\nabla^\perp$-parallel isotropic subbundle Ξ. Consider the quotient bundle $\mathcal{S} := \mathrm{Coker}(\Xi \hookrightarrow \Xi^\perp)$ and choose a non-canonical realization $S \subset NX$ inducing a splitting $NX = \Xi \oplus S \oplus \Theta$ as in Cor. 2.6. Consider the tube $\mathcal{U}_S \subset \mathbb{R}^{r+s}$ along S in $\mathbb{R}^{r,s}$ and write $\tilde{f} : \mathcal{U}_S \to \mathbb{R}^{r+s}$ for the embedding. Hence, $\mathcal{U}_S$ has codimension 2 and $\mathfrak{hol}(\nabla^{\perp,\tilde{f}}) = 0$ since we suppose that $\tilde{f}$ has flat normal bundle. There is an open covering $X = \bigcup_k X_k$ and nowhere vanishing sections $V^k \in \Gamma(X_k, \Xi|_{X_k})$. Write $\tilde{f}_k : \mathcal{U}_S^k \to \mathbb{R}^{r+s}$ for the tube of X_k along $S|_{X_k}$. Thus, $\tilde{f}_k$ has flat normal bundle. Let $p \in X_k$ and $\alpha^w : (-\varepsilon, +\varepsilon) \to X_k$ such that $\alpha^w(0) = p$ and $\dot{\alpha}^w(0) = w$. For any local section $V_s \in \Gamma(\alpha^w, S)$ such that $V_0 \in S_p \cap U_k$ and $v \in \Xi_p$ we have

$$\langle v, \underbrace{w - A^f_{V_0}(w)}_{\in T_pX_k} + \underbrace{\nabla^{\perp,f}_w V_s}_{\in \Xi_p^\perp}\rangle = 0,$$

i.e., $\tau^{\nabla^{\mathbb{R}^{r,s}}}_{\gamma_{V_0}}(v) \in N_{\exp^\perp(V_0)}\mathcal{U}_S^k$. Let $q = \exp^\perp(V_0)$ and define $\tilde{V}_q := \tau^{\nabla^{\mathbb{R}^{r,s}}}_{\gamma_{V_0}}(V_p^k)$ as well as $\tilde{\Xi}_q := \mathrm{span}\{\tilde{V}_q\}$. This way we derive a section $\tilde{V} \in \Gamma(\mathcal{U}_S^k, N\mathcal{U}_S^k)$ such that $\tilde{V}|_{X_k} = V^k$ and a subbundle $\tilde{\Xi} \subset N\mathcal{U}_S^k$. Since $\mathfrak{hol}(\nabla^{\perp,\tilde{f}_k}) \subset \mathfrak{so}(1,1)$ we have $\nabla^{\perp,\tilde{f}_k}_{\cdot}\tilde{V} \in \mathbb{R} \cdot \tilde{V}$. Since $\tilde{f}_k$ has flat normal bundle we have w.l.o.g. $h_k \in C^\infty(\mathcal{U}_S^k)$ such that $\nabla^{\perp,\tilde{f}_k}_{\cdot}(h_k\tilde{V}) = 0$. Thus, X admits an open covering with local $\nabla^{\perp,f}$-parallel sections $h_k|_{X_k}V^k$ and the same proof as in Lemma 2.14.3 implies the statement. $\square$

Thm. 2.11 and Thm. 3.14 imply

Theorem 3.19. *Let $f : X \to Y$ be a spacelike submanifold in a Lorentzian space of constant curvature and $p \in X$. Suppose the Borel-Lichnérowicz decomposition of N_pX is given by*

$$N_pX = E_0 \oplus E_1 \oplus \ldots \oplus E_\ell \quad \textit{and} \quad \mathfrak{hol}_p(\nabla^\perp) = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_\ell$$

where $\mathfrak{h}_1$ acts weakly irreducibly with index 1 on E_1.

Then for $j \geq 2$ each $\mathfrak{h}_j$ acts on E_j as the holonomy representation of a Riemannian symmetric space and $\mathfrak{h}_1 \subset \mathfrak{so}(1, q+1)$ is given by one of the following

- *Type 1:* $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^q$
- *Type 2:* $\mathfrak{h} = \mathfrak{g} \ltimes \mathbb{R}^q$
- *Type 3:*
$$\mathfrak{h} = \left\{ \begin{pmatrix} \varphi(A) & w^T & 0 \\ 0 & A & -w \\ 0 & 0 & -\varphi(A) \end{pmatrix} : A \in \mathfrak{g},\ w \in \mathbb{R}^q \right\}$$
where $\varphi : \mathfrak{g} \twoheadrightarrow \mathbb{R}$ is an epimorphism satisfying $\varphi|_{[\mathfrak{g},\mathfrak{g}]} = 0$.
- *Type 4: There is $0 < \ell < q$ such that $\mathbb{R}^q = \mathbb{R}^\ell \oplus \mathbb{R}^{q-\ell}$, $\mathfrak{g} \subset \mathfrak{so}(\ell)$ and*
$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & \psi(A)^T & w^T & 0 \\ 0 & 0 & 0 & -\psi(A) \\ 0 & 0 & A & -w \\ 0 & 0 & 0 & 0 \end{pmatrix} : A \in \mathfrak{g},\ w \in \mathbb{R}^\ell \right\}$$
for some epimorphism $\psi : \mathfrak{g} \twoheadrightarrow \mathbb{R}^{q-\ell}$ satisfying $\psi|_{[\mathfrak{g},\mathfrak{g}]} = 0$,

where $\mathfrak{g}$ acts as the holonomy representation of a Riemannian symmetric space. □

Remark 3.20. The algebra $\mathfrak{g}$ in Thm. 3.19 has a Borel-Lichnérowicz decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ where each $\mathfrak{g}_i$ acts as the holonomy representation of an irreducible Riemannian symmetric space.

Moreover, if $\mathfrak{g}$ is given by type 3 or 4 then some $\mathfrak{g}_j$ has non-trivial center, i.e., $\mathfrak{g}_j$ acts as the holonomy representation of an irreducible Hermitian symmetric space (cf. [BCO03][Appx. A4]). □

Next, we consider very good submanifolds with non-degenerately reducible normal screen holonomy representation.

Proposition 3.21. *Let $f : (X, g) \to (\mathbb{R}^{r+s}, \langle\cdot,\cdot\rangle_{r,s})$ be a simply connected embedded very good submanifold whose normal holonomy acts weakly irreducibly with index 1. Let S be a non-canonical realization of the normal screen bundle and $\mathfrak{g}$ its holonomy representation at $p \in X$.*

If $S_p = S_1|_p \oplus S_2|_p$ is a $\mathfrak{g}$-invariant orthogonal decomposition into non-degenerate subspaces such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_i$ acts trivially on $S_j|_p$ for $i \neq j$ and $\mathfrak{hol}_p(\nabla^{\perp,f})$ contains the ideal $\mathbb{R}^{\operatorname{codim} X - 2}$, i.e.,

$$\left\{ \begin{pmatrix} 0 & w^T & 0 \\ 0 & 0 & -w \\ 0 & 0 & 0 \end{pmatrix} : w \in \mathbb{R}^{\operatorname{codim} X - 2} \right\} \subset \mathfrak{hol}_p(\nabla^{\perp,f}) \subset \mathfrak{so}(S_p, \langle\cdot,\cdot\rangle) \ltimes \mathbb{R}^{\operatorname{codim} X - 2}$$

then

- *there is an open subset* $\tilde{X} \supset X$ *of* $\mathcal{U}_{S_2} \subset (\mathbb{R}^{r+s}, \langle\cdot,\cdot\rangle_{r,s})$ *which is a very good submanifold, where* S_2 *is the subbundle corresponding to* $S_2|_p$*,*
- $\mathfrak{hol}_p(\nabla^{\perp,\tilde{X}})$ *acts weakly irreducibly with index 1,*
- *the normal screen bundle of* $\tilde{X}$ *extends the bundle* S_1 *corresponding to* $S_1|_p$ *and the normal screen holonomy of* $\nabla^{\perp,\tilde{X}}$ *is given by* $\mathfrak{g}_1$*.*

Proof. We write $\tilde{f} : \mathcal{U}_{S_2} \to (\mathbb{R}^{r+s}, \langle\cdot,\cdot\rangle_{r,s})$ for the embedding of $\mathcal{U}_{S_2}$. For $w \in T_pX$ and $V \in S_2|_p$ let $\alpha_s^{F_V^{-1}(w)} : (-\varepsilon, +\varepsilon) \to X$ such that $\alpha_0^{F_V^{-1}(w)} = p$ and $\dot{\alpha}_0^{F_V^{-1}(w)} = F_V^{-1}(w)$. Let $V_s \in \Gamma(\alpha^{F_V^{-1}(w)}, S_2)$ such that $V_0 = V$ and define $\beta_s := \exp^{\perp}(V_s)$. We already know

$$\dot{\beta}_0 = \tau_{\gamma_V}^{\nabla^{\mathbb{R}^{r,s}}}(w + \underbrace{\nabla^{\perp,f}_{F_V^{-1}(w)} V_s}_{\in S_2 \oplus \Xi}) \in T_{\beta_0}\mathcal{U}_{S_2}.$$

Moreover, for any $\tilde{V} \in S_2|_p$ we may define $V_s := V + s\tilde{V}$ along the constant curve $\alpha_s^{F_V^{-1}(0)} = p$. Hence, $\tau_{\gamma_V}^{\nabla^{\mathbb{R}^{r,s}}}(\tilde{V}) \in T_{\beta_0}\mathcal{U}_{S_2}$ for all $\tilde{V} \in S_2|_p$. Let $(w_1, \dots, w_{\dim X})$ be a pseudo-orthonormal basis of (T_pX, g_p). For any w_i we can find $V_s \in \Gamma(\alpha^{F_V^{-1}(w_i)}, S_2)$ such that $pr_{S_2} \circ \nabla^{\perp,f}_{F_V^{-1}(w_i)} V_s = 0$ and $V_0 = V$. Since $S_2|_p$ is invariant w.r.t. $(pr_S \circ \nabla^{\perp,f}_{\cdot})$ we have $pr_S \circ \nabla^{\perp,f}_{F_V^{-1}(w_i)} V_s = 0$, i.e.,

$$\tau_{\gamma_V}^{\nabla^{\mathbb{R}^{r,s}}}(w_i + \underbrace{\nabla^{\perp,f}_{F_V^{-1}(w_i)} V_s}_{\in \Xi}) \in T_{\exp^{\perp}(V_0)}\mathcal{U}_{S_2}.$$

Therefore, $\tau_{\gamma_V}^{\nabla^{\mathbb{R}^{r,s}}}(v), \tau_{\gamma_V}^{\nabla^{\mathbb{R}^{r,s}}}(e) \in N_{\exp^{\perp}(V_0)}\mathcal{U}_{S_2}$ for all $v \in \Xi_p, e \in S_1|_p$. We define an isotropic subbundle $\tilde{\Xi} \subset N\mathcal{U}_{S_2}$ of rank 1 by $\tilde{\Xi}_{\beta_0} := \tau_{\gamma_V}^{\nabla^{\mathbb{R}^{r,s}}}(\Xi_p)$.

Next, we show $\nabla^{\perp,\tilde{f}}_{\cdot}\tilde{\Xi} \subset \tilde{\Xi}$. Consider the curve $\beta_t : [0,1] \to \mathcal{U}_{S_2}$ and its projection $\alpha_t : [0,1] \to X$. For $V_t \in \Gamma(\alpha_t, S_2)$ such that $\beta_t = \exp^{\perp}(V_t)$ and $\tilde{\xi}_0 \in \tilde{\Xi}_{\beta_0}$ let $\xi_t \in \Gamma(\alpha_t, \Xi)$ be the $\nabla^{\perp,f}$-parallel vector field along α_t such that $\tilde{\xi}_0 = \tau_{\gamma_{V_0}}^{\nabla^{\mathbb{R}^{r,s}}}\xi_0$. We define $\tilde{\xi}_t \in \Gamma(\beta_t, \tilde{\Xi})$ by $\tilde{\xi}_t := \tau_{\gamma_{V_t}}^{\nabla^{\mathbb{R}^{r,s}}}\xi_t$. Hence, we have to prove $\nabla^{\perp,\tilde{f}}_{\dot{\beta}_t}\tilde{\xi}_{\cdot} \in \Gamma(\beta_t, \tilde{\Xi})$. Using the standard coordinate vector fields $\partial_1, \dots, \partial_{r+s}$ of $\mathbb{R}^{r+s}$ we have $\xi_t = \xi_t^i \partial_i|_{\alpha_t}$ and therefore $\tilde{\xi}_t = \xi_t^i \partial_i|_{\beta_t}$. Thus,

$$\begin{aligned} \nabla^{\mathbb{R}^{r,s}}_{\dot{\beta}_t}\tilde{\xi}_{\cdot} = \dot{\xi}_t^i \partial_i|_{\beta_t} &= \tau_{\gamma_{V_t}}^{\nabla^{\mathbb{R}^{r,s}}}(\dot{\xi}_t^i \partial_i|_{\alpha_t}) = \tau_{\gamma_{V_t}}^{\nabla^{\mathbb{R}^{r,s}}}(\nabla^{\mathbb{R}^{r,s}}_{\dot{\alpha}_t}\xi_{\cdot}) \\ &= \tau_{\gamma_{V_t}}^{\nabla^{\mathbb{R}^{r,s}}}(-A^f_{\xi_t}(\dot{\alpha}_t) + \nabla^{\perp,f}_{\dot{\alpha}_t}\xi_{\cdot}) \\ &= -\tau_{\gamma_{V_t}}^{\nabla^{\mathbb{R}^{r,s}}}(A^f_{\xi_t}(\dot{\alpha}_t)), \end{aligned}$$

and all we need to show is $pr_{N_{\beta_t}\mathcal{U}_{S_2}}(\tau_{\gamma_{V_t}}^{\nabla^{\mathbb{R}^{r,s}}}(T_{\alpha_t}X)) \in \tilde{\Xi}_{\beta_t}$. However, as we have already seen, for $w \in T_{\alpha_t}X$ and some appropriate $v \in \Xi_{\alpha_t}$ we have

$$\tau_{\gamma_{V_t}}^{\nabla^{\mathbb{R}^{r,s}}}(w) = \underbrace{\tau_{\gamma_{V_t}}^{\nabla^{\mathbb{R}^{r,s}}}(w+v)}_{\in T_{\exp^\perp(V_t)}\mathcal{U}_{S_2}} - \underbrace{\tau_{\gamma_{V_t}}^{\nabla^{\mathbb{R}^{r,s}}}(v)}_{\in \tilde{\Xi}_{\exp^\perp(V_t)}}.$$

Since $\tau_{\gamma_V}^{\nabla^{\mathbb{R}^{r,s}}}(e) \in N_{\exp^\perp(V_0)}\mathcal{U}_{S_2}$ for all $e \in S_1|_p$ we define

$$\tilde{S}_{\beta_0} := \operatorname{span}\{\tau_{\gamma_V}^{\nabla^{\mathbb{R}^{r,s}}}(e) : e \in S_1|_p\} \subset N_{\exp^\perp(V_0)}\mathcal{U}_{S_2}.$$

Therefore, we derive a subbundle $\tilde{S} \subset \tilde{\Xi}^\perp \subset N\mathcal{U}_{S_2}$ on $\mathcal{U}_{S_2}$ such that $\tilde{S}_p = S_1|_p$ for all $p \in X \subset \mathcal{U}_{S_2}$.

In order to compute the holonomy of $(pr_{\tilde{S}} \circ \nabla^{\perp,\tilde{f}}_{\cdot}|_{\tilde{S}})$ we consider again the curves α_t and $\beta_t = \exp^\perp(V_t)$. For any $\tilde{\eta}_0 \in \tilde{S}_{\beta_0}$ let $\eta_t \in \Gamma(\alpha_t, S_1)$ be the $(pr_S \circ \nabla^{\perp,f}_{\cdot}|_S)$-parallel vector field along α_t such that $\tilde{\eta}_0 = \tau_{\gamma_{V_0}}^{\nabla^{\mathbb{R}^{r,s}}}\eta_0$. We define $\tilde{\eta}_t \in \Gamma(\beta_t, \tilde{S})$ by $\tilde{\eta}_t := \tau_{\gamma_{V_t}}^{\nabla^{\mathbb{R}^{r,s}}}\eta_t$. As above we conclude

$$\begin{aligned}
\nabla^{\perp,\tilde{f}}_{\dot{\beta}_t}\tilde{\eta}_\cdot &= pr_{N_{\beta_t}\mathcal{U}_{S_2}}(\nabla^{\mathbb{R}^{r,s}}_{\dot{\beta}_t}\tilde{\eta}_\cdot) \\
&= pr_{N_{\beta_t}\mathcal{U}_{S_2}}(\tau_{\gamma_{V_t}}^{\nabla^{\mathbb{R}^{r,s}}}(-A^f_{\eta_t}(\dot{\alpha}_t) + \nabla^{\perp,f}_{\dot{\alpha}_t}\eta_\cdot)) \\
&= -\underbrace{pr_{N_{\beta_t}\mathcal{U}_{S_2}}(\tau_{\gamma_{V_t}}^{\nabla^{\mathbb{R}^{r,s}}}(A^f_{\eta_t}(\dot{\alpha}_t)))}_{\in \tilde{\Xi}_{\beta_t}} + pr_{N_{\beta_t}\mathcal{U}_{S_2}}(\underbrace{\tau_{\gamma_{V_t}}^{\nabla^{\mathbb{R}^{r,s}}}(pr_\Xi \circ \nabla^{\perp,f}_{\dot{\alpha}_t}\eta_\cdot)}_{\in \tilde{\Xi}_{\beta_t}}),
\end{aligned}$$

i.e., $\tilde{\eta}_t$ is $(pr_{\tilde{S}} \circ \nabla^{\perp,\tilde{f}}_{\cdot}|_{\tilde{S}})$-parallel along β_t. Hence, $\mathfrak{g}_1 = \mathfrak{hol}_p(pr_{\tilde{S}} \circ \nabla^{\perp,\tilde{f}}_{\cdot}|_{\tilde{S}})$.

Now, we construct $\tilde{X} \subset X$ using the same approach as in Proposition 3.16. Let $(E_1, \ldots, E_m)$ be a local pseudo-orthonormal frame for S_1 around $p \in X$. If $w \in T_pX$ we have

$$\begin{aligned}
A^{\mathcal{U}_{S_2}}_{E_k}(w) &= -pr_{T_p\mathcal{U}_{S_2}} \circ \nabla^{\mathbb{R}^{r,s}}_w E_k \\
&= -pr_{T_p\mathcal{U}_{S_2}}(-\underbrace{A^f_{E_k}(w)}_{\in T_pX \subset T_p\mathcal{U}_{S_2}} + \underbrace{\nabla^{\perp,f}_w E_k}_{\in S_1 \subset N_p\mathcal{U}_{S_2}}) \\
&= A^f_{E_k}(w).
\end{aligned}$$

Consider a pseudo-orthonormal basis $(s_1, \ldots, s_d)$ of $S_2|_p$ and a pseudo-orthonormal basis $(e_1, \ldots, e_n)$ of T_pX. We extend each $E_k|_p$ to a section $\tilde{E}_k \in \Gamma(\gamma_{s_i}, S_1)$ along the geodesic γ_{s_i} with $\gamma_{s_i}(0) = p$ and $\dot{\gamma}_{s_i}(0) = s_i$ by $\tilde{E}_k|_{\gamma_{s_i}(t)} := \tau_{\gamma_{s_i}(t)}^{\nabla^{\mathbb{R}^{r,s}}}(E_k) \in \tilde{S}_{\gamma_{s_i}(t)}$. Hence, $A^{\mathcal{U}_{S_2}}_{E_k}(s_i) = -pr_{T_p\mathcal{U}_{S_2}} \circ \nabla^{\mathbb{R}^{r,s}}_{s_i}\tilde{E}_k = 0$ and the matrix of $A^{\mathcal{U}_{S_2}}_{E_k}$ in the basis $(s_1, \ldots, s_d, e_1, \ldots, e_n)$ of $T_p\mathcal{U}_{S_2}$ has the form

$$A^{\mathcal{U}_{S_2}}_{E_k} = \begin{pmatrix} 0 & 0 \\ 0 & A^f_{E_k} \end{pmatrix}.$$

We conclude

$$[A^{\mathcal{U}_{S_2}}_{E_k}, A^{\mathcal{U}_{S_2}}_{E_\ell}] = \begin{pmatrix} 0 & 0 \\ 0 & [A^f_{E_k}, A^f_{E_\ell}] \end{pmatrix}$$

and

$$Tr([A^{\mathcal{U}_{S_2}}_{E_k}, A^{\mathcal{U}_{S_2}}_{E_\ell}] \cdot [A^{\mathcal{U}_{S_2}}_{E_{\bar{k}}}, A^{\mathcal{U}_{S_2}}_{E_{\bar{\ell}}}]) = Tr([A^f_{E_k}, A^f_{E_\ell}] \cdot [A^f_{E_{\bar{k}}}, A^f_{E_{\bar{\ell}}}]).$$

By continuity there is an open subset $X \subset \tilde{X} \subset \mathcal{U}_{S_2}$ such that $\tilde{X}$ is a very good submanifold. Since $X \subset \tilde{X}$ all statements on the holonomy which we have proved for $\mathcal{U}_{S_2}$ remain true once we consider the restriction of $\tilde{S}$ and $\tilde{\Xi}$ to $\tilde{X}$.

So far, we have not shown $\mathfrak{hol}_p(\nabla^{\perp,\tilde{X}})$ to act weakly irreducibly. However, once we have shown that $\mathfrak{hol}_p(\nabla^{\perp,\tilde{X}})$ contains the ideal $\mathbb{R}^{\operatorname{codim}\tilde{X}-2}$ we may conclude $\mathfrak{hol}_p(\nabla^{\perp,\tilde{X}})$ to act weakly irreducibly with index 1 having the screen holonomy $\mathfrak{g}_1$. Define $r = \operatorname{codim}\tilde{X} - 2$. Clearly, $\mathfrak{hol}_p(\nabla^{\perp,f})$ contains the ideal $\mathbb{R}^r$ if and only if $Hol^0_p(\nabla^{\perp,f})$ contains the closed subgroup $\mathbb{R}^r$ having the matrix form

$$\left\{ \begin{pmatrix} 1 & -Y^T & -\frac{1}{2}Y^TY \\ 0 & 1_{r\times r} & Y \\ 0 & 0 & 1 \end{pmatrix} : Y \in \mathbb{R}^r \right\}$$

w.r.t. the basis $(v, t_1, \dots, t_{r-d}, s_1, \dots, s_d, z)$ where $\langle v, z\rangle = 1$ and $(t_1, \dots, t_{r-d})$ is a pseudo-orthonormal basis of $S_1|_p$. Hence, there are loops $\gamma_i : [0,1] \to X$ around p for $1 \le i \le r-d$ such that

$$\tau^{\perp,f}_{\gamma_i} = \begin{pmatrix} 1 & -E_i^T & -\frac{1}{2}Y^TY \\ 0 & 1_{r\times r} & E_i \\ 0 & 0 & 1 \end{pmatrix}$$

where $\{E_1, \dots, E_r\}$ is the standard basis of $\mathbb{R}^r$. However, for loops in X we have $\tau^{\perp,f}_{\gamma_i}|_{S_1|_p} = \tau^{\perp,\tilde{X}}_{\gamma_i}$, i.e.,

$$\begin{pmatrix} 1 & -E_i^T & -\frac{1}{2}Y^TY \\ 0 & 1_{r\times r} & E_i \\ 0 & 0 & 1 \end{pmatrix} \in Hol^0_p(\nabla^{\perp,\tilde{X}}), \qquad \forall 1 \le i \le r-d.$$

Hence, $\mathfrak{hol}_p(\nabla^{\perp,\tilde{X}})$ contains the ideal $\mathbb{R}^{\operatorname{codim}\tilde{X}-2}$. □

Remark 3.22. In Prop. 3.21 we imposed the condition $\mathbb{R}^{\operatorname{codim}X-2} \subset \mathfrak{hol}(\nabla^{\perp,f})$ on the holonomy of $\nabla^{\perp,f}$ to insure $\mathfrak{hol}(\nabla^{\perp,\tilde{X}})$ not leaving a non-degenerate subspace invariant. If the screen bundle is not positive definite we cannot apply Thm. 2.11 to understand how restrictive this condition is. However, for spacelike submanifolds in Minkowski space Thm. 3.19 implies that $\mathfrak{hol}(\nabla^{\perp,\tilde{X}})$ acts weakly irreducibly if $\mathfrak{hol}(\nabla^{\perp,f})$ is not of type 4.

In order to prove the proposition if f is only an immersion we consider the immersion of its universal covering space $\check{X}$. In this case, the proof works once we construct $\mathcal{U}_{S_2}$ using an immersed tubular neighborhood of $\check{X}$. □

Finally, we consider very good submanifolds with non-degenerately reducible normal holonomy representation.

Proposition 3.23. *Let $f : (X, g) \to (\mathbb{R}^{r+s}, \langle\cdot,\cdot\rangle_{r,s})$ be a simply connected embedded very good submanifold whose Borel-Lichnérowicz decomposition is given by*

$$N_pX = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \ldots \oplus \mathcal{E}_\ell \quad \textit{and} \quad \mathfrak{hol}_p(\nabla^\perp) = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_\ell.$$

Then for any $0 \le i \le \ell$

- *there is an open subset $\tilde{X} \supset X$ of $\mathcal{U}_{E_i^\perp} \subset (\mathbb{R}^{r+s}, \langle\cdot,\cdot\rangle_{r,s})$ which is a very good submanifold, where each E_j is the subbundle corresponding to $\mathcal{E}_j$,*
- *$\mathfrak{hol}_p(\nabla^{\perp,\tilde{X}}) = \mathfrak{h}_i$,*
- *the normal bundle of $\tilde{X}$ extends E_i.*

Proof. Write $\tilde{f} : \mathcal{U}_{E_i^\perp} \to (\mathbb{R}^{r+s}, \langle\cdot,\cdot\rangle_{r,s})$ for the embedding of $\mathcal{U}_{E_i^\perp}$. For $q \in X$, $w \in T_qX$ and $V \in E_i^\perp|_q$ let $\alpha_s^{F_V^{-1}(w)} : (-\varepsilon, +\varepsilon) \to X$ such that $\alpha_0^{F_V^{-1}(w)} = q$ and $\dot{\alpha}_0^{F_V^{-1}(w)} = F_V^{-1}(w)$. Moreover, let $V_s \in \Gamma(\alpha^{F_V^{-1}(w)}, E_i^\perp)$ such that $V_0 = V$ and define $\beta_s = \exp^\perp(V_s)$. Thus,

$$\dot{\beta}_0 = \tau_{\gamma_V}^{\nabla^{\mathbb{R}^{r,s}}}(w + \nabla^{\perp,f}_{F_V^{-1}(w)} V_s) \in T_{\beta_0}\mathcal{U}_{E_i^\perp}.$$

On the other hand, if we let $\tilde{V} \in E_i^\perp|_q$ and define $\tilde{V}_s := V + s\tilde{V}$ on the constant curve $\alpha_s^{F_V^{-1}(0)} = q$ then for $\tilde{\beta}_s := \exp^\perp(\tilde{V}_s)$ we conclude $\beta_0 = \tilde{\beta}_0$ and $\dot{\tilde{\beta}}_0 = \tau_{\gamma_V}^{\nabla^{\mathbb{R}^{r,s}}}(\tilde{V}) \in T_{\beta_0}\mathcal{U}_{E_i^\perp}$. Hence,

$$T_{\beta_0}\mathcal{U}_{E_i^\perp} = \tau_{\gamma_V}^{\nabla^{\mathbb{R}^{r,s}}}(T_qX \oplus E_i^\perp|_q) \quad \text{and} \quad N_{\beta_0}\mathcal{U}_{E_i^\perp} = \tau_{\gamma_V}^{\nabla^{\mathbb{R}^{r,s}}}(E_i|_q).$$

Suppose $\beta_t : [0,1] \to \mathcal{U}_{E_i^\perp}$ is a smooth curve having the projection $\alpha_t : [0,1] \to X$. Let $V_t \in \Gamma(\alpha_t, E_i^\perp)$ such that $\beta_t = \exp^\perp(V_t)$. For any $\tilde{\xi}_0 \in N_{\beta_0}\mathcal{U}_{E_i^\perp}$ let $\xi_t \in \Gamma(\alpha_t, E_i)$ be the $\nabla^{\perp,f}$-parallel vector field along α_t such that $\tilde{\xi}_0 = \tau_{\gamma_{V_0}}^{\mathbb{R}^{r,s}}(\xi_0)$. Define $\tilde{\xi}_t \in \Gamma(\beta_t, N\mathcal{U}_{E_i^\perp})$ by $\tilde{\xi}_t := \tau_{\gamma_{V_t}}^{\nabla^{\mathbb{R}^{r,s}}}(\xi_t)$. Using the standard coordinate vector fields $\partial_1, \ldots, \partial_{r+s}$ of $\mathbb{R}^{r+s}$ once again we have $\xi_t = \xi_t^i \partial_i|_{\alpha_t}$ and therefore $\tilde{\xi}_t = \xi_t^i \partial_i|_{\beta_t}$, i.e.,

$$\begin{aligned}
\nabla^{\mathbb{R}^{r,s}}_{\dot{\beta}_t} \tilde{\xi}_\cdot &= \dot{\xi}_t^i \partial_i|_{\beta_t} = \tau_{\gamma_{V_t}}^{\nabla^{\mathbb{R}^{r,s}}}(\dot{\xi}_t^i \partial_i|_{\alpha_t}) = \tau_{\gamma_{V_t}}^{\nabla^{\mathbb{R}^{r,s}}}(\nabla^{\mathbb{R}^{r,s}}_{\dot{\alpha}_t} \xi_\cdot) \\
&= \tau_{\gamma_{V_t}}^{\nabla^{\mathbb{R}^{r,s}}}(-A^f_{\xi_t}(\dot{\alpha}_t) + \nabla^{\perp,f}_{\dot{\alpha}_t} \xi_\cdot) \\
&= -\tau_{\gamma_{V_t}}^{\nabla^{\mathbb{R}^{r,s}}}\underbrace{(A^f_{\xi_t}(\dot{\alpha}_t))}_{\in T_{\alpha_t}X} \in T_{\beta_t}\mathcal{U}_{E_i^\perp}.
\end{aligned}$$

Hence, $\tilde{\xi}_t$ is $\nabla^{\perp,\tilde{f}}$-parallel along β_t and it remains to show the good submanifold property. For a curve $\beta_s : (-\varepsilon, +\varepsilon) \to \mathcal{U}_{E_i^\perp}$ let $\alpha_s : (-\varepsilon, +\varepsilon) \to X$ be its projection. Define $q := \alpha_0$ and $V_s \in \Gamma(\alpha_s, E_i^\perp)$ such that $\beta_s = \exp^\perp(V_s)$. If $(e_1, \ldots, e_{\dim \mathcal{E}_i})$ is a pseudo-orthonormal basis of $E_i|_q$ then $\tilde{e}_k := \tau_{\gamma_{V_0}}^{\nabla^{\mathbb{R}^{r,s}}}(e_k)$ induces a pseudo-orthonormal basis of $N_{\beta_0}\mathcal{U}_{E_i^\perp}$.

Consider the $\nabla^{\perp,\tilde{f}}$-parallel vector field $\tilde{E}_k(s)$ along β_s such that $\tilde{E}_k(0) = \tilde{e}_k$. Thus, $\tilde{E}_k(s) = \tau^{\nabla^{\mathbb{R}^{r,s}}}_{\gamma_{V_s}}(\check{E}_k(s))$, where $\check{E}_k(s)$ is the $\nabla^{\perp,f}$-parallel vector field along α_s such that $\check{E}_k(0) = e_k$. We conclude

$$A^{\tilde{f}}_{\tilde{e}_k}(\dot{\beta}_0) = \tau^{\nabla^{\mathbb{R}^{r,s}}}_{\gamma_{V_0}}(A^f_{e_k}(\dot{\alpha}_0)).$$

If $\beta_0 \in X$ this implies $Tr([A^{\tilde{f}}_{\tilde{e}_k}, A^{\tilde{f}}_{\tilde{e}_\ell}] \cdot [A^{\tilde{f}}_{\tilde{e}_k}, A^{\tilde{f}}_{\tilde{e}_\ell}]) = Tr([A^f_{e_k}, A^f_{e_\ell}] \cdot [A^f_{e_k}, A^f_{e_\ell}])$ at β_0. By continuity there is an open subset $\tilde{X} \supset X$ of $\mathcal{U}_{E_i^\perp}$ such that $\tilde{X}$ is a very good submanifold of $(\mathbb{R}^{r+s}, \langle \cdot, \cdot \rangle_{r,s})$ and this implies the statement. $\square$

Appendix A: Walker Coordinates and Suitable Functions

In this section we assume all greek indices to be contained in $\{1,\dots,n\}$ and all latin indices to be contained in $\{0,\dots,n+1\}$. Let $(U,\tilde{g})$ be a local coordinate neighborhood given by $(x^0,x^1,\dots,x^n,x^{n+1})=(x,y^1,\dots,y^n,z)$ with a Lorentzian metric $\tilde{g}$ of the form

$$\tilde{g}=2dxdz+2u_\alpha dy^\alpha dz+g_{\alpha\beta}dy^\alpha dy^\beta+fdz^2,$$

where $u_\alpha, f, g_{\alpha\beta}\in C^\infty(U)$ and $\frac{\partial u_\alpha}{\partial x}=\frac{\partial g_{\alpha\beta}}{\partial x}=0$. Moreover, $g_{\alpha\beta}$ is a family of Riemannian metrics on the submanifolds $U_{c_1c_2}:=\{(c_1,y^1,\dots,y^n,c_2)\}$, where $c_1,c_2\in\mathbb{R}$. Hence, the matrix of $\tilde{g}$ and its inverse $\tilde{g}^{-1}$ are given by

$$\tilde{g}=\begin{pmatrix}0&0&\cdots&0&1\\0&&&&u_1\\\vdots&&g_{\alpha\beta}&&\vdots\\0&&&&u_n\\1&u_1&\cdots&u_n&f\end{pmatrix}\quad \tilde{g}^{-1}=\begin{pmatrix}(g^{\alpha\beta}u_\alpha u_\beta-f)&-g^{\alpha 1}u_\alpha&\cdots&-g^{\alpha n}u_\alpha&1\\-g^{1\beta}u_\beta&&&&0\\\vdots&&g^{\alpha\beta}&&\vdots\\-g^{n\beta}u_\beta&&&&0\\1&0&\cdots&0&0\end{pmatrix},$$

where $(g^{\alpha\beta})$ is the inverse of $(g_{\alpha\beta})$. In these coordinates the Christoffel symbols $\tilde{\Gamma}^k_{ij}$ of $\tilde{g}$ are given by $\tilde{\Gamma}^k_{ij}=\frac{1}{2}\tilde{g}^{k\ell}(\partial_i\tilde{g}_{j\ell}+\partial_j\tilde{g}_{i\ell}-\partial_\ell\tilde{g}_{ij})$. We compute

$$\begin{aligned}\tilde{\Gamma}^k_{0i}&=\frac{1}{2}\tilde{g}^{k\ell}(\partial_0\tilde{g}_{i\ell}+\underbrace{\partial_i\tilde{g}_{0\ell}}_{=0}-\underbrace{\partial_\ell\tilde{g}_{0i}}_{=0})\\&=\frac{1}{2}(\tilde{g}^{k0}\underbrace{\partial_0\tilde{g}_{i0}}_{=0}+\tilde{g}^{k(n+1)}\partial_0\tilde{g}_{i(n+1)}+\tilde{g}^{k\alpha}\underbrace{\partial_0\tilde{g}_{i\alpha}}_{=0})=\frac{1}{2}\delta_{k0}\delta_{i(n+1)}\frac{\partial f}{\partial x}\text{ and}\end{aligned}$$

$$\begin{aligned}
\tilde{\Gamma}^k_{(n+1)\alpha} &= \frac{1}{2}\tilde{g}^{k\ell}(\partial_{n+1}\tilde{g}_{\alpha\ell} + \partial_\alpha \tilde{g}_{(n+1)\ell} - \partial_\ell \tilde{g}_{(n+1)\alpha}) \\
&= \frac{1}{2}\tilde{g}^{k\ell}\partial_{n+1}\tilde{g}_{\alpha\ell} + \frac{1}{2}\tilde{g}^{k0}(\partial_{n+1}\underbrace{\tilde{g}_{\alpha 0}}_{=0} - \underbrace{\partial_0 \tilde{g}_{(n+1)\alpha}}_{=0}) \\
&\quad + \frac{1}{2}\tilde{g}^{k\beta}(\partial_\alpha \tilde{g}_{(n+1)\beta} - \partial_\beta \tilde{g}_{(n+1)\alpha}) \\
&\quad + \frac{1}{2}\tilde{g}^{k(n+1)}(\partial_\alpha \tilde{g}_{(n+1)(n+1)} - \partial_{n+1}\tilde{g}_{(n+1)\alpha}) \\
&= \frac{1}{2}\tilde{g}^{k\ell}\partial_{n+1}\tilde{g}_{\alpha\ell} + \frac{1}{2}\tilde{g}^{k\beta}(\partial_\alpha u_\beta - \partial_\beta u_\alpha) + \frac{1}{2}\delta_{k0}(\frac{\partial f}{\partial y^\alpha} - \frac{\partial u_\alpha}{\partial z}) \\
&= \frac{1}{2}\tilde{g}^{k0}\partial_{n+1}\underbrace{\tilde{g}_{\alpha 0}}_{=0} + \frac{1}{2}\tilde{g}^{k\beta}\partial_{n+1}\tilde{g}_{\alpha\beta} + \frac{1}{2}\tilde{g}^{k(n+1)}\partial_{n+1}\tilde{g}_{\alpha(n+1)} \\
&\quad + \frac{1}{2}\tilde{g}^{k\beta}(\partial_\alpha u_\beta - \partial_\beta u_\alpha) + \frac{1}{2}\delta_{k0}(\frac{\partial f}{\partial y^\alpha} - \frac{\partial u_\alpha}{\partial z}) \\
&= \frac{1}{2}\tilde{g}^{k\beta}(\partial_\alpha u_\beta - \partial_\beta u_\alpha + \frac{\partial g_{\alpha\beta}}{\partial z}) + \frac{1}{2}\delta_{k0}\frac{\partial f}{\partial y^\alpha} \\
&= \frac{1}{2}\delta_{k\gamma}g^{\gamma\beta}(\partial_\alpha u_\beta - \partial_\beta u_\alpha + \frac{\partial g_{\alpha\beta}}{\partial z}) \\
&\quad + \frac{1}{2}\delta_{k0}(\frac{\partial f}{\partial y^\alpha} - g^{\beta\gamma}u_\gamma(\partial_\alpha u_\beta - \partial_\beta u_\alpha + \frac{\partial g_{\alpha\beta}}{\partial z})) \text{ as well as}
\end{aligned}$$

$$\begin{aligned}
\tilde{\Gamma}^k_{(n+1)(n+1)} &= \frac{1}{2}\tilde{g}^{k\ell}(2\partial_{n+1}\tilde{g}_{(n+1)\ell} - \partial_\ell \tilde{g}_{(n+1)(n+1)}) \\
&= -\frac{1}{2}\tilde{g}^{k0}\frac{\partial f}{\partial x} + \frac{1}{2}\tilde{g}^{k\alpha}(2\frac{u_\alpha}{\partial z} - \frac{\partial f}{\partial y^\alpha}) + \frac{1}{2}\delta_{k0}\frac{\partial f}{\partial z} \\
&= \frac{1}{2}\delta_{k\beta}g^{\beta\alpha}(2\frac{u_\alpha}{\partial z} - \frac{\partial f}{\partial y^\alpha} - u_\alpha\frac{\partial f}{\partial x}) + \frac{1}{2}\delta_{k(n+1)}\frac{\partial f}{\partial x} \\
&\quad + \frac{1}{2}\delta_{k0}((g^{\alpha\beta}u_\alpha u_\beta - f)\frac{\partial f}{\partial x} - g^{\alpha\beta}u_\beta(2\frac{u_\alpha}{\partial z} - \frac{\partial f}{\partial y^\alpha}) + \frac{\partial f}{\partial z}).
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\tilde{\Gamma}^k_{\alpha\beta} &= \frac{1}{2}\tilde{g}^{k\ell}(\partial_\alpha \tilde{g}_{\beta\ell} + \partial_\beta \tilde{g}_{\alpha\ell} - \partial_\ell \tilde{g}_{\alpha\beta}) \\
&= \frac{1}{2}\tilde{g}^{k0}\underbrace{(\partial_\alpha \tilde{g}_{\beta 0} + \partial_\beta \tilde{g}_{\alpha 0} - \partial_0 \tilde{g}_{\alpha\beta})}_{=0} + \frac{1}{2}\tilde{g}^{k\gamma}(\partial_\alpha \tilde{g}_{\beta\gamma} + \partial_\beta \tilde{g}_{\alpha\gamma} - \partial_\gamma \tilde{g}_{\alpha\beta}) \\
&\quad + \frac{1}{2}\tilde{g}^{k(n+1)}(\partial_\alpha \tilde{g}_{\beta(n+1)} + \partial_\beta \tilde{g}_{\alpha(n+1)} - \partial_{n+1}\tilde{g}_{\alpha\beta}) \\
&= \delta_{k\lambda}\Gamma^\lambda_{\alpha\beta} + \frac{1}{2}\delta_{k0}(\frac{\partial u_\beta}{\partial y^\alpha} + \frac{\partial u_\alpha}{\partial y^\beta} - \frac{\partial g_{\alpha\beta}}{\partial z} - 2\Gamma^\lambda_{\alpha\beta}u_\lambda).
\end{aligned}$$

For $Y_\alpha := \partial_\alpha - u_\alpha \partial_0$ let $S := \mathrm{span}\{Y_1, \ldots, Y_n\}$ be a non-canonical realization of the screen bundle. Then $\Theta = \mathrm{span}\{Z\}$ where $Z := \partial_{n+1} - \frac{1}{2} f \partial_0$ and $\Xi = \mathrm{span}\{\partial_0\}$. In particular, $\tilde{g}(Y_\alpha, Y_\beta) = g_{\alpha\beta}$ and

$$[Y_\alpha, Y_\beta] = (\partial_\beta u_\alpha - \partial_\alpha u_\beta)\partial_0, \qquad [Z, Y_\alpha] = (\frac{1}{2}\partial_\alpha f - \partial_{n+1} u_\alpha)\partial_0,$$
$$[Z, \partial_0] = \frac{1}{2}(\partial_0 f)\partial_0, \qquad [Y_\alpha, \partial_0] = 0.$$

As we have seen above ∂_0 is a recurrent lightlike vector field in $(U, \tilde{g})$. We are interested in the case in which $\mathfrak{hol}(U, \tilde{g})$ is weakly irreducible.

Definition 4.1. *Let $(U, \tilde{g})$ be given Walker coordinates of the form*

$$\tilde{g} = 2dxdz + 2u_\alpha dy^\alpha dz + g_{\alpha\beta} dy^\alpha dy^\beta + f dz^2.$$

We say that $f \in C^\infty(U)$ is suitable if $\mathfrak{hol}(U, \tilde{g})$ is weakly irreducible and not of type 4 in Thm. 2.11.

Lemma 4.2. *Let $\mathfrak{h} \subset \mathfrak{so}(1, n+1)$ have the Borel-Lichnérowicz property and $v \in \mathbb{R}^{1,n+1}$ such that $Hv \in \mathbb{R} \cdot v$ for all $H \in \mathfrak{h}$ and $\langle v, v \rangle = 0$. Let $(v, y_1, \ldots, y_n, z)$ be a basis of $\mathbb{R}^{1,n+1}$ such that $\langle v, z \rangle = 1$, $\langle v, y_i \rangle = \langle z, y_i \rangle = \langle z, z \rangle = 0$ and $\langle y_i, y_j \rangle = \delta_{ij}$. Suppose there are $H_1, \ldots, H_n \in \mathfrak{h}$ such that $\langle H_i y_j, z \rangle = \delta_{ij}$ and $H_i v = 0$. If $S := span\{y_1, \ldots, y_n\}$ and $\mathfrak{g} := span\{pr_S \circ H|_S : H \in \mathfrak{h}\} \subset \mathfrak{so}(n)$ is trivial or irreducible then $\mathfrak{h}$ is weakly irreducible with index 1 and not of type 4.*

Proof. Suppose $w \in \mathbb{R}^{1,n+1}$ is timelike and $\mathfrak{h}$-invariant. Since $0 = \langle Hw, w \rangle + \langle w, Hw \rangle$ we may assume $\mathfrak{h} \cdot w = 0$ and $\langle w, w \rangle = -1$. Write $w = \alpha v + \beta^i y_i + \gamma z$. Then $\langle w, w \rangle = -1$ implies $\gamma \neq 0$. For any $H \in \mathfrak{h}$ we have $Hv = \mu_H v$ and

$$\begin{aligned} 0 &= Hw \\ &= \alpha \mu_H v + \sum_{j=1}^n \beta^i \langle H y_i, y_j \rangle y_j + \beta^i \langle H y_i, z \rangle v + \gamma \langle Hz, v \rangle z + \gamma \sum_{j=1}^n \langle Hz, y_j \rangle y_j \\ &= (\alpha \mu_H + \beta^i \langle H y_i, z \rangle) v + \sum_{j=1}^n \langle H(\beta^i y_i + z), y_j \rangle y_j + \gamma \langle Hz, v \rangle z. \end{aligned}$$

Hence, $0 = \gamma \langle Hz, v \rangle = -\gamma \langle z, Hv \rangle = \gamma \mu_H$, i.e., $\mu_H = 0$ for all $H \in \mathfrak{h}$ and $0 = \alpha \mu_H + \beta^i \langle H y_i, z \rangle = \beta^i \langle H y_i, z \rangle$. Since $\mathbb{R}^n \subset \{(\langle H y_1, z \rangle, \ldots, \langle H y_n, z \rangle) : H \in \mathfrak{h}\}$ we can choose $H_k \in \mathfrak{h}$ such that $\langle H_k y_i, z \rangle = \delta_{ik}$. Thus, $\beta^k = 0$ for all $k \in \{1, \ldots, n\}$ and $\langle w, w \rangle = -1$ implies $w = \alpha v - \frac{1}{2\alpha} z$ for some $\alpha \neq 0$. Since $\mathfrak{h} w = 0$ we conclude $\mathfrak{h} z = 0$ impliying the contradiction $0 = H_k z = \sum_{j=1}^n \langle H_k z, y_j \rangle y_j = -y_k$.

Assume $\mathfrak{h}$ admits an invariant non-degenerate subspace $E \subset \mathbb{R}^{1,n+1}$. W.l.o.g. we may assume that E is timelike not containing a non-degenerate invariant subspace and $\dim E \geq 2$. In this case, the Borel-Lichnérowicz property implies $\mathfrak{h} = \mathfrak{h}|_E \oplus \mathfrak{h}|_{E^\perp}$ where $\mathfrak{h}|_E \subset \mathfrak{so}(1, \dim E - 1)$ and $\mathfrak{h}|_{E^\perp} \subset \mathfrak{so}(\dim E^\perp)$. There is no lightlike $\mathfrak{h}$-invariant line if

$\mathfrak{h}|_E$ is irreducible. Therefore, $\mathfrak{h}|_E$ is weakly irreducible with index 1 and we have $\tilde{v} \in E$ such that $\mathfrak{h} \cdot \tilde{v} \subset \mathbb{R} \cdot \tilde{v}$ and $\langle \tilde{v}, \tilde{v} \rangle = 0$. If $v \notin E$ we have $v = t + s$ with $t \in E$ and $s \in E^\perp$. Then $\mathfrak{h}v \in \mathbb{R}v$ implies $\mathfrak{h}t \in \mathbb{R}t$ and we derive an invariant timelike vector. If $\tilde{v}, v \in E$ are linearly independent we have w.l.o.g. $\langle v, \tilde{v} \rangle = 1$, and $\mu_H = \langle Hv, \tilde{v} \rangle = -\langle v, H\tilde{v} \rangle$ implies $H\tilde{v} = -\mu_H \tilde{v}$. Thus, $E = \mathrm{span}\{v, \tilde{v}\}$ since $\mathrm{span}\{\frac{1}{2}(v - \tilde{v}), \frac{1}{2}(v + \tilde{v})\} \subset E$ is non-degenerate and $\mathfrak{h}$-invariant. Moreover, $\tilde{v} = z + \beta^i y_i + \gamma v$ for some $\beta^1, \ldots, \beta^n \in \mathbb{R}$ and $\gamma = -\frac{1}{2} \sum_i (\beta^i)^2$. For $i \in \{1, \ldots, n\}$ we conclude $H_i \tilde{v} = -\mu_{H_i} \tilde{v} = 0$ and

$$0 = \langle H_i \tilde{v}, z \rangle = \langle H_i z, z \rangle + \beta^j \langle H_i y_j, z \rangle + \gamma \langle H_i v, z \rangle = \beta^j \langle H_i y_j, z \rangle + \gamma \mu_{H_i} = \beta^j \delta_{ij},$$

i.e., $\tilde{v} = z$. However, this implies $S = E^\perp$ and the contradiction $\delta_{ij} = \langle H_i y_j, z \rangle = 0$ since $H_i y_j \in E^\perp$ by the Borel-Lichnérowicz property.

Therefore, $\tilde{v} \in \mathbb{R} \cdot v$. If $\dim E = 2$ then there is $\tilde{z} \in E$ such that $\|\tilde{z}\| = 0$, $\langle v, \tilde{z} \rangle = 1$. We have $H\tilde{z} \in E$ by the Borel-Lichnérowicz property as well as $\langle H\tilde{z}, \tilde{z} \rangle = 0$ and $\langle H\tilde{z}, v \rangle = -\mu_H$. Hence, $H\tilde{z} \in \mathbb{R}\tilde{z}$ for all $H \in \mathfrak{h}$ and we derive a contradiction as above.

Thus, $\dim E > 2$ and $S \cap (\mathbb{R} \cdot v)^\perp \cap E \neq 0$. Suppose $E^\perp \neq 0$. Then there is a basis $w_1, \ldots, w_n$ of S such that $w_i - \lambda_i v \in E^\perp$ for $1 \leq i \leq \dim E^\perp$ and $w_i \in E$ for $i > \dim E^\perp$. Thus, the Borel-Lichnérowicz property implies that $\mathrm{span}\{w_1, \ldots, w_{\dim E^\perp}\}$ is $\mathfrak{g}$-invariant. In particular, $\mathfrak{g}$ is not irreducible and the statement follows from Thm. 2.11 unless $\mathfrak{g} = 0$. However, if $\mathfrak{g} = 0$ and $w_1 = \alpha^j y_j$ then $H_i(w_1 - \lambda_1 v) = \alpha^j H_i y_j = \alpha^j \langle H_i y_j, z \rangle v = \alpha^j \delta_{ij} v$ and we derive a contradiction since $\alpha^j \neq 0$ for some j, i.e., $0 \neq H_i(w_1 - \lambda_1 v) \in E$. □

Observe that the statement of the Lemma remains true if we replace the orthonormal basis $(y_1, \ldots, y_n)$ by any basis of $\mathrm{span}\{v, z\}^\perp$. If for some $p \in U$ and any $\alpha \in \{1, \ldots, n\}$ we can find $Y \in \mathrm{span}\{V, Z\}^\perp$ such that $\tilde{g}(R_p(Y, Z)Y_\beta, Z) = \delta_{\alpha\beta}$ then $\mathfrak{hol}_p(U, \tilde{g})$ is weakly irreducible by Lemma 4.2 and the Ambrose-Singer Theorem unless $\mathfrak{g}$ is non-trivial and reducible. Define

$$F_{\alpha\beta} := \partial_z(\Gamma^0_{\alpha\beta}) - \partial_z(\Gamma^\gamma_{\alpha\beta})u_\gamma + \partial_\alpha(\omega^\gamma_\beta)u_\gamma + \frac{1}{2}\partial_\alpha(u_\gamma)\omega^\gamma_\beta + \frac{1}{2}\omega^\gamma_\beta \Gamma^\delta_{\alpha\gamma} u_\delta - \frac{1}{2}\omega^\gamma_\beta \Gamma^0_{\alpha\gamma},$$

where $\omega^\alpha_\beta := g^{\alpha\gamma}(\partial_\beta u_\gamma - \partial_\gamma u_\beta + \frac{g_{\gamma\beta}}{\partial z})$.[6]

Lemma 4.3. *Suppose $(U, \tilde{g})$ are Walker coordinates of the form*

$$\tilde{g} = 2dxdz + 2u_\alpha dy^\alpha dz + g_{\alpha\beta} dy^\alpha dy^\beta + f dz^2$$

such that $x(p) = y^\alpha(p) = 0$. If $\mathfrak{g} \subset \mathfrak{so}(n)$ is trivial or irreducible then $\mathfrak{hol}(U, \tilde{g})$ is weakly irreducible and ∂_0 is

- *parallel if $f = \frac{1}{2}(F_{\alpha\alpha} + 1) \sum_\alpha (y^\alpha)^2 + \sum_{\alpha \neq \beta} \frac{F_{\alpha\beta}}{2} y^\alpha y^\beta$,*
- *recurrent but not parallel if $f = \frac{1}{2}(F_{\alpha\alpha} + x^2 + 1) \sum_\alpha (y^\alpha)^2 + \sum_{\alpha \neq \beta} \frac{F_{\alpha\beta}}{2} y^\alpha y^\beta$.*

[6] Notice, that no term in the definition of $F_{\alpha\beta}$ includes a term involving f.

Proof. A long computation shows

$$\begin{aligned}\tilde{g}(R_p(Y_\alpha, Z)Y_\beta, Z) = &\frac{\partial^2 f}{\partial y^\alpha \partial y^\beta} \\ &- \frac{1}{2}(\Gamma^\gamma_{\alpha\beta}\frac{\partial f}{\partial y^\gamma} + u_\alpha \frac{\partial f}{\partial x \partial y^\beta} + u_\beta \frac{\partial f}{\partial x \partial y^\alpha} - u_\alpha u_\beta \frac{\partial^2 f}{\partial x^2}) \\ &- F_{\alpha\beta}.\end{aligned}$$

Given our choices for f we conclude $\tilde{g}(R_p(Y_\alpha, Z)Y_\beta, Z) = \delta_{\alpha\beta}$ since $y^\alpha(p) = 0$ for all $\alpha \in \{1, \ldots, n\}$. In order to apply Lemma 4.2 we have to show $\tilde{g}(R_p(Y_\alpha, Z)V, Z) = 0$. We compute

$$\begin{aligned}\tilde{g}(R_p(Y_\alpha, Z)V, Z) &= \tilde{g}(R_p(\partial_\alpha, \partial_{n+1})\partial_0, \partial_{n+1}) - u_\alpha \tilde{g}(R_p(\partial_0, \partial_{n+1})\partial_0, \partial_{n+1}) \\ &= \frac{1}{2}\frac{\partial^2 f}{\partial x \partial y^\alpha}|_p - u_\alpha \frac{1}{2}\frac{\partial^2 f}{\partial x^2}|_p\end{aligned}$$

However, $x(p) = y^\alpha(p) = 0$ implies $\frac{\partial^2 f}{\partial x \partial y^\alpha}|_p = \frac{\partial^2 f}{\partial x^2}|_p = 0$. □

By Lemma 4.2 $\mathfrak{hol}(U, \tilde{g})$ is of type 2 if $f \in C^\infty(U)$ is given by the first case. On the other hand, if f is given by the second case then $\mathfrak{hol}(U, \tilde{g})$ is of type 1 or 3 since $\tilde{g}(R_q(\partial_0, \partial_{n+1})\partial_0, \partial_{n+1}) \neq 0$ at $q \in U$ with $x(q), y^\alpha(q) \neq 0$, i.e., ∂_0 cannot be rescaled to a parallel vector field.

Appendix B: The Killing Form on so(r,s)

When studying the normal holonomy of submanifolds in pseudo-Riemannian spaces of constant curvature we will obtain several classification results for submanifolds whose kernel of the normal curvature tensor respects a certain condition. In order to state this condition we need to introduce a scalar product on $\Lambda^2\mathbb{R}^{r,s}$. Remember, that the Lie group of linear isometries of $\mathbb{R}^{r,s}$ can be identified with

$$O(r,s) := \{g \in GL(r+s,\mathbb{R}) : g^T = I_{r,s}g^{-1}I_{r,s}\}, \quad I_{r,s} := \begin{pmatrix} -1_{r\times r} & 0 \\ 0 & 1_{s\times s} \end{pmatrix}$$

and the Lie algebra of $O(r,s)$ is given by

$$\begin{aligned} \mathfrak{so}(r,s) &:= \{A \in \mathfrak{gl}(r+s,\mathbb{R}) : A^T I_{r,s} + I_{r,s}A = 0\} \\ &= \{A \in \mathfrak{gl}(r+s,\mathbb{R}) : \langle Av, w\rangle_{r,s} = -\langle v, Aw\rangle_{r,s},\ v, w \in \mathbb{R}^{r+s}\}. \end{aligned}$$

Consider the following map

$$\begin{aligned} F : \Lambda^2\mathbb{R}^{r+s} &\to \mathfrak{so}(r,s) \\ v \wedge w &\mapsto F(v\wedge w), \end{aligned}$$

where $\langle F(v\wedge w)x, y\rangle_{r,s} := \langle v, x\rangle_{r,s}\langle w, y\rangle_{r,s} - \langle w, x\rangle_{r,s}\langle v, y\rangle_{r,s}$. Then F is invertible and if $e_1, \ldots, e_{r+s}$ is a pseudo-orthonormal basis of $\mathbb{R}^{r,s}$ its inverse map is given by $F^{-1}(A) = \sum_{i<j} \varepsilon_i\varepsilon_j\langle A(e_i), e_j\rangle_{r,s} e_i \wedge e_j$. On the Lie algebra $\mathfrak{so}(r,s)$ we have the symmetric bilinear form

$$\kappa : \mathfrak{so}(r,s) \times \mathfrak{so}(r,s) \to \mathbb{R}, \qquad (A,B) \mapsto \frac{1}{2}Tr(A\cdot B)$$

and an induced symmetric bilinear form $B(\cdot,\cdot) := \kappa(F(\cdot), F(\cdot))$ on $\Lambda^2\mathbb{R}^{p+q}$. Since

$$\langle F(e_i\wedge e_j)e_k, e_\ell\rangle = \varepsilon_k\varepsilon_\ell(\delta_{ik}\delta_{j\ell} - \delta_{jk}\delta_{i\ell})$$

we compute for $i < j$ and $k < \ell$

$$\begin{aligned}
B(e_i \wedge e_j, e_k \wedge e_\ell) &= \kappa(F(e_i \wedge e_j), F(e_k \wedge e_\ell)) \\
&= \frac{1}{2} Tr(F(e_i \wedge e_j) \cdot F(e_k \wedge e_\ell)) \\
&= \frac{1}{2} \sum_{\alpha=1}^{r+s} (\sum_{\beta=1}^{r+s} \varepsilon_\alpha \langle F(e_i \wedge e_j) e_\beta, e_\alpha \rangle \varepsilon_\beta \langle F(e_k \wedge e_\ell) e_\alpha, e_\beta \rangle) \\
&= \frac{1}{2} \sum_{\alpha,\beta=1}^{r+s} \varepsilon_\alpha \varepsilon_\beta (\delta_{i\beta}\delta_{j\alpha} - \delta_{j\beta}\delta_{i\alpha})(\delta_{k\alpha}\delta_{\ell\beta} - \delta_{\ell\alpha}\delta_{k\beta}) \\
&= \frac{1}{2} \sum_{\alpha,\beta=1}^{r+s} \varepsilon_\alpha \varepsilon_\beta (\delta_{i\beta}\delta_{j\alpha}\delta_{k\alpha}\delta_{\ell\beta} + \delta_{j\beta}\delta_{i\alpha}\delta_{\ell\alpha}\delta_{k\beta} \\
&\qquad - \delta_{j\beta}\delta_{i\alpha}\delta_{k\alpha}\delta_{\ell\beta} - \delta_{i\beta}\delta_{j\alpha}\delta_{\ell\alpha}\delta_{k\beta}) \\
&= \varepsilon_i \varepsilon_j (\delta_{jk}\delta_{i\ell} - \delta_{ik}\delta_{j\ell}) \\
&= -\varepsilon_i \varepsilon_j \delta_{ik}\delta_{j\ell}.
\end{aligned}$$

Hence, $\{e_i \wedge e_j : 1 \leq i < j \leq r+s\}$ is a pseudo-orthonormal basis of $(\Lambda^2\mathbb{R}^{r+s}, B)$ and B has signature $(\frac{(r+s)(r+s-1)}{2} - rs, rs)$. Let $ad_A(B) := [A, B]$ be the adjoint representation of $\mathfrak{so}(r,s)$ and consider the Killing form $Tr(ad_A \circ ad_B)$. Since

$$\begin{aligned}
[F(e_i \wedge e_j), F(e_k \wedge e_\ell)] &= \varepsilon_i \delta_{i\ell} F(e_j \wedge e_k) + \varepsilon_j \delta_{jk} F(e_i \wedge e_\ell) \\
&\quad + \varepsilon_j \delta_{j\ell} F(e_k \wedge e_i) + \varepsilon_i \delta_{ki} F(e_\ell \wedge e_j)
\end{aligned}$$

we conclude

$$\begin{aligned}
[F(e_\alpha \wedge e_\beta), [F(e_i \wedge e_j), F(e_k \wedge e_\ell)]] &= (\varepsilon_i\delta_{i\ell}\varepsilon_\alpha\delta_{\alpha k} - \varepsilon_i\delta_{ki}\varepsilon_\alpha\delta_{\ell\alpha})F(e_\beta \wedge e_j) \\
&\quad + (\varepsilon_i\delta_{i\ell}\varepsilon_\beta\delta_{\beta j} - \varepsilon_j\delta_{j\ell}\varepsilon_\beta\delta_{\beta i})F(e_\alpha \wedge e_k) \\
&\quad + (\varepsilon_i\delta_{ki}\varepsilon_\beta\delta_{\beta\ell} - \varepsilon_i\delta_{i\ell}\varepsilon_\beta\delta_{\beta k})F(e_\alpha \wedge e_j) \\
&\quad + (\varepsilon_j\delta_{j\ell}\varepsilon_\alpha\delta_{\alpha i} - \varepsilon_i\delta_{i\ell}\varepsilon_\alpha\delta_{j\alpha})F(e_\beta \wedge e_k) \\
&\quad + (\varepsilon_j\delta_{jk}\varepsilon_\alpha\delta_{\alpha\ell} - \varepsilon_j\delta_{j\ell}\varepsilon_\alpha\delta_{k\alpha})F(e_\beta \wedge e_i) \\
&\quad + (\varepsilon_j\delta_{jk}\varepsilon_\beta\delta_{\beta i} - \varepsilon_i\delta_{ki}\varepsilon_\beta\delta_{\beta j})F(e_\alpha \wedge e_\ell) \\
&\quad + (\varepsilon_j\delta_{j\ell}\varepsilon_\beta\delta_{\beta k} - \varepsilon_j\delta_{jk}\varepsilon_\beta\delta_{\beta\ell})F(e_\alpha \wedge e_i) \\
&\quad + (\varepsilon_i\delta_{ki}\varepsilon_\alpha\delta_{\alpha j} - \varepsilon_j\delta_{jk}\varepsilon_\alpha\delta_{i\alpha})F(e_\beta \wedge e_\ell).
\end{aligned}$$

Using $\alpha < \beta$, $i < j$ and $k < \ell$ we derive

$$\begin{aligned}
pr_{F(e_k\wedge e_\ell)}([F(e_\alpha \wedge e_\beta),&[F(e_i \wedge e_j), F(e_k \wedge e_\ell)]]) \\
&= (-\delta_{jk}\delta_{\beta\ell} + \delta_{\beta k}\delta_{j\ell})(\varepsilon_i\delta_{i\ell}\varepsilon_\alpha\delta_{\alpha k} - \varepsilon_i\delta_{ki}\varepsilon_\alpha\delta_{\ell\alpha}) \\
&\quad - \delta_{\alpha\ell}(\varepsilon_i\delta_{i\ell}\varepsilon_\beta\delta_{\beta j} - \varepsilon_j\delta_{j\ell}\varepsilon_\beta\delta_{\beta i}) \\
&\quad + (-\delta_{jk}\delta_{\alpha\ell} + \delta_{\alpha k}\delta_{j\ell})(\varepsilon_i\delta_{ki}\varepsilon_\beta\delta_{\beta\ell} - \varepsilon_i\delta_{i\ell}\varepsilon_\beta\delta_{\beta k}) \\
&\quad - \delta_{\beta\ell}(\varepsilon_j\delta_{j\ell}\varepsilon_\alpha\delta_{\alpha i} - \varepsilon_i\delta_{i\ell}\varepsilon_\alpha\delta_{j\alpha}) \\
&\quad + (\delta_{ik}\delta_{\beta\ell} + \delta_{\beta k}\delta_{i\ell})(\varepsilon_j\delta_{jk}\varepsilon_\alpha\delta_{\alpha\ell} - \varepsilon_j\delta_{j\ell}\varepsilon_\alpha\delta_{k\alpha}) \\
&\quad + \delta_{\alpha k}(\varepsilon_j\delta_{jk}\varepsilon_\beta\delta_{\beta i} - \varepsilon_i\delta_{ki}\varepsilon_\beta\delta_{\beta j}) \\
&\quad + (-\delta_{ik}\delta_{\alpha\ell} + \delta_{\alpha k}\delta_{i\ell})(\varepsilon_j\delta_{j\ell}\varepsilon_\beta\delta_{\beta k} - \varepsilon_j\delta_{jk}\varepsilon_\beta\delta_{\beta\ell}) \\
&\quad + \delta_{\beta k}(\varepsilon_i\delta_{ki}\varepsilon_\alpha\delta_{\alpha j} - \varepsilon_j\delta_{jk}\varepsilon_\alpha\delta_{i\alpha}) \\
&= \varepsilon_\alpha\varepsilon_\beta\delta_{\alpha i}\delta_{\beta j}(-\delta_{i\ell} - \delta_{j\ell} - \delta_{ki} - \delta_{jk} + 2\delta_{ki}\delta_{j\ell}).
\end{aligned}$$

Finally, we have $\sum_{k<\ell}(-\delta_{i\ell} - \delta_{j\ell} - \delta_{ki} - \delta_{jk} + 2\delta_{ki}\delta_{j\ell}) = -(i-1) - (j-1) - (r+s-i) - (r+s-j) + 2$ and therefore

$$\begin{aligned}
Tr(ad_{F(e_\alpha\wedge e_\beta)} \circ ad_{F(e_i\wedge e_j)}) &= -2(r+s-2)\varepsilon_\alpha\varepsilon_\beta\delta_{\alpha i}\delta_{\beta j} \\
&= 2(r+s-2)B(e_\alpha \wedge e_\beta, e_i \wedge e_j),
\end{aligned}$$

In particular, a subspace $W \subset \Lambda^2\mathbb{R}^{r+s}$ is non-degenerate w.r.t. B if and only if $F(W)$ is non-degenerate w.r.t. the Killing form on $\mathfrak{so}(r,s)$.

This observation will be applied in order to define *(very) good* submanifolds for which $W := (\mathrm{Ker}(R^\perp))^\perp$ has to be (definite) non-degenerate w.r.t. the Killing form on $\Lambda^2\mathbb{R}^{r,s}$.

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Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 23.02.2011 Kordian Lärz